THE

MATHEMATICAL GAZETTE

EDITED BY

T. A. A. BROADBENT, M.A.

ROYAL NAVAL COLLEGE, GREENWICH, LONDON, S.E. 10.

LONDON

G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY

VOL. XXVIII. FEBRUARY, 1944. No. 278

GENERAL MEETING: APRIL 1944.

AT a meeting on 8th January, the Executive Committee, acting for the Council, decided to hold a General Meeting of the Association on Wednesday, 12th April, and Thursday, 13th April, at King's College, Strand, London, W.C. 2. It will open with a short business meeting at 11 a.m. on Wednesday.

Although it is not possible to give full details of the programme at present, it may be stated that two important discussions will be held. The first of these will concern possible changes in mathematics at the School Certificate level and may include discussion of the proposals of the sub-committee mentioned in the December *Gazette*. The subject-matter of the second discussion will be a syllabus for mathematics at the Sixth Form stage for science students. This has been prepared by the Science Masters Association, working in conjunction with various members of the Mathematical Association. Details are printed in this number of the *Gazette*, pp. i–iv. It is hoped that a large number of members will make an effort to attend these important discussions.

The Programme Committee wishes to thank those members who offered to speak or sent suggestions. It has not proved possible to make use of all the offers and suggestions, but a careful record will be kept of these as they may be suitable for use on a future occasion. G. L. P.

ON THE HARMONIC PROPERTIES OF TWO CONICS.

BY H. GWYNEDD GREEN.

THE textbook demonstrations of the well-known properties of two conics, the common self-polar triangle and the conic-envelope of lines cut in harmonic conjugates by them, usually suffer from the defects that the general case only is given and that the methods are not always adaptable to special cases. As the author feels the habit of contentment with general cases only a dangerous one and encouraging of loose thinking, he proposes to list the special cases, indicating for the sake of brevity proofs in outline only or, where very trivial, omitting them entirely. A third property—that of the conic-locus of points at which the conics subtend harmonic conjugates, as the immediate reciprocal of the second, has not been listed.

Notation.

The conics will be denoted by S_1, S_2 and their four points of intersection by A, B, C, D: the meet of the joins of AC, BD will be denoted by X, that of AD, BC by Y, and that of AB, CD by Z. Except where expressly stated otherwise it will be supposed that the conics are non-degenerate and that the points A, B, C, D are general. The trivial case in which a conic degenerates to one straight line counted twice has been omitted.

The self-polar property.

If a point P has the same polar line for the two conics, it follows immediately that the join of P to one of the four points A, B, C, D must pass through a second. The only possible positions of P are therefore X, Y, Z and, from the harmonic properties of the quadrangle, the corresponding polar lines common to S_1 and S_2 are YZ, ZX and XY, and we have the unique self-polar triangle XYZ.

Special cases.

(1) A, B coincide, simple contact: for P, the point (AB) and Z at the intersection of the tangent there with CD.

(2) A, B coincide and C, D coincide, double contact: for P, any pair of points on the chord of contact harmonically separated by the points of contact, and Z at the pole of this chord.

(3) A, B, C coincide, three-point contact: for P, the point (ABC).

(4) A, B, C, D coincide, four-point contact: for P, any point on the common tangent.

In cases (5) to (8) S_1 degenerates to the line pair AB, CD.

(5) The points general in position: for P, X at AC, BD and Y at BC, AD.

(6) A, B coincide, AB a tangent: for P, the point (AB).

(7) A, B coincide and C, D coincide: for P, any pair of points on the chord of contact harmonically separated by the points of contact.

(8) A, B, C coincide, a tangent and chord through the point of contact: no position for P.

In cases (9), (10) S_1 and S_2 degenerate to the line pairs AB, CD and AD, BC.

(9) The points general in position: for P, X at AC, BD.

(10) A, B coinciding, the lines of one pair intersecting on a line of the other: no position for P.

The envelope of harmonically divided chords.

We project the general figure by taking C, D as the circular points. Let P, Q be the centres of the circles of radii r_1, r_2 thus obtained and L, M the feet of their perpendiculars on any line divided harmonically by the circles, as in the figure.

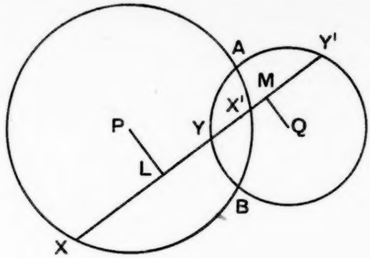
Since L is the mid-point of XX' and $\{XX', YY'\} = -1$,

$$r_1^2 - PL^2 = LX^2 = LY \, . \, LY' = LQ^2 - r_2^2$$

$$\text{or} \quad PL^2 + QL^2 = r_1^2 + r_2^2.$$

The locus of L is therefore a circle whose centre is the mid-point of PQ, and hence the envelope of the line is a conic with foci P and Q. In the original figure it follows that the envelope is a conic touching the eight tangents to S_1, S_2 through their points of intersection.*

* An interesting alternative construction for the envelope is as follows: If p, the polar of P on S_1 with respect to S_2, meets S_1 at R, S, then PR, PS are the tangents to the envelope from P.



Special cases.

(1) If the pole of AB with respect to S_1 coincides with the pole of CD with respect to S_2, the circles in the projected figure are orthogonal. We have immediately the well-known theorem : " If the poles of AB with respect to S_1 and CD with respect to S_2 coincide, so also do the poles of AB with respect to S_2 and CD with respect to S_1 ", and also " The conic envelope degenerates to these two poles."

(2) A, B coincide, simple contact : no material modification.

(3) A, B coincide, and C, D coincide, double contact : projecting these points to the circular points, the circles obtained are concentric with a concentric circle envelope, the conic envelope has double contact with S_1, S_2.

(4) A, B, C coincide (three-point contact) and A, B, C, D coincide (four-point contact) : the basic construction as given fails. Projecting one conic into a circle, we can take the equations as $x^2+y^2+2fy=0$,

$$x^2+y^2+2fy+y(px+qy)=0,$$

and the envelope is $2f^2l^2+4fm-fpl-2-q=0$, with p zero in the case of four-point contact.

In cases (5) to (10) S_1 degenerates to the line pair AB, CD.

(5) The points in general position : the four common tangents of the envelope and S_1 become the lines themselves, the points of contact being the intersections of XY with the line pair. In the projected figure the conic is a parabola.

(6) AB, CD are conjugate lines with respect to S_2 : the envelope degenerates to points on the lines.

(7) A, B coincide : no material modification.

(8) B, D coincide : projecting A, C to the circular points the chord becomes a chord of S_2 either subtending a right angle at (BD) or through (BD), or in the original figure the envelope is the pole of AC and the point (BD).

(9) A, B coincide and C, D coincide, two tangents : the envelope becomes a conic touching the lines at the points of contact.

(10) A, B, C coincide, tangent and chord through the point of contact : the envelope becomes the pole of AD and the point (ABC).

If S_1 and S_2 both degenerate to line pairs,

(11) In the general case the envelope becomes a conic touching them.

(12) If the point of intersection of S_1 lies on a line of S_2, the envelope becomes that point and a point on the other line of S_2 (the conjugate of the intersection of the S_2 lines with its other intersections with S_1). H. G. G.

COPERNICUS—NEWTON—EINSTEIN.*

By T. Arnold Brown.

The remarkable coincidence that the fourth centenary of the death of Nicholas Copernicus in May, 1543, as well as the third centenaries of the death of Galileo in January, 1642, and the birth of Isaac Newton in December, 1642, all fall within the present academic session has induced me to present a rapid survey of the history of mathematical life and thought, with particular emphasis on cosmology, from the first stirrings of the Renaissance down to the present day.

Copernicus belongs to that rare class of creative thinkers who, combining a high degree of moral courage with a superb integrity of intellect, are impelled to call in question the conventional beliefs of their day, sanctified though they be by the full weight of established authority and by long years of tradition, prejudice or superstition. He is therefore well worthy to rank with those inspired seekers after truth who appear with unfailing regularity in the history of mathematics and who—if we may borrow a striking phrase of Einstein's—achieve immortality by " challenging an axiom ". Moreover, the work of Copernicus in his day was destined to pave the way for the dazzling achievements of Newton in the century which followed.

Nicholas Copernicus was born in February, 1473, in the little trading centre of Thorn (or should it be Torun?) on the Vistula; the town lay in a region over which the King of Poland exercised some form of authority; but, even at that early stage in the chequered history of Europe, it did not go unchallenged by the Order of Teutonic Knights.

Though he completed early in life the first draft of the book which was to make his name so famous, Copernicus shrank from publication and devoted himself to revising and rewriting the manuscript, partly from a complete indifference to personal fame and even more from a distaste for the controversy to which the publication of such original ideas was bound to give rise. He was a loyal son of Mother Church, and the extent to which he challenged the prevailing ideas of his age was exceeded only by the discretion which he displayed in avoiding polemics and dispute during his life-time.

At the age of thirty-nine he entered upon a Canonry at Frauenburg, to which he had been earlier appointed, and while continuing his work in mathematics and astronomy he became a busy man of affairs, much preoccupied with the business of the Chapter. In 1521 he was commissioned to draw up a statement of the grievances of the Chapter against the Teutonic Knights for presentation to the Prussian Estates, and in the following year wrote a memorandum on the debased and confused state of the coinage in the district. In these respects, as in his reluctance to publish and aversion to controversy, there may be traced a curious parallel between his life and that of Newton who, in his later years, became a Member of Parliament, a man of affairs and, as Master of the Mint, the guardian of his country's coinage.

In the year 1539 Copernicus was visited by an enthusiastic young astronomer generally known as Rheticus, who held one of the mathematical chairs at the Protestant University at Wittenberg. The visit extended over nearly two years, during which time Rheticus set himself to study the manuscript of Copernicus. When he eventually returned to Wittenberg, it had probably been already settled that he was to perform the service which Roger Cotes was later to undertake in regard to the second edition of Newton's *Principia*, namely, superintend the printing of the complete book itself.

* Being the substance of a lecture delivered to the Plymouth Branch of the Association at University College, Exeter, on Saturday, May 1st, 1943.

But Rheticus was not able to see it all through the press himself and the work was entrusted to a Lutheran preacher—Andreas Osiander—who did what he could to mitigate the heresy which characterised the book by adding a preface in which he asserted that the fundamental ideas laid down in it were merely abstract hypotheses convenient for purposes of calculation. He also gave it the deceptively innocent title *De Revolutionibus Orbium Celestium* —the last two words being probably his own addition.

It was not until the last days of Copernicus' life that a printed copy of the precious volume was placed in his hands.

The central idea with which the memory of Copernicus is associated is that all velocities are relative to some observer and that the apparent motions of the celestial bodies are to a great extent an illusion due to the motion of this mortal globe carrying the observer with it. Before Copernicus, there was one chief "System of the Physical World"—the Ptolemaic—in which the earth was accorded a privileged position at the centre of the universe. Henceforth there were two. More than a century elapsed before the new idea received general acceptance, and more than three centuries were to pass before men realised that not only was the derivative $\frac{ds}{dt}$ a purely relative conception, but likewise the s and the t themselves.

Isaac Newton was born on Christmas Day, 1642, in the first sad winter of the Civil War, at Woolsthorpe Manor House, which is situated near the Great North Road some six miles south of Grantham and a few minutes from the Parish Church of Colsterworth, where the entry of his baptism on January 1, 1643, may still be seen. The room in which the birth took place bears a tablet over the mantelpiece recording the event and quoting the famous lines of Pope.

His early education was received at the King's School, Grantham, where, after an interruption of about three years devoted to work on the family farm, he eventually prepared for entry to Trinity College, Cambridge.

He was not expected to walk the six miles to school each day, and so he lodged at the house of an apothecary in High Street, Grantham. There he unearthed a parcel of old books, including a number on alchemy, which remained a subject of interest throughout his life. He also fell in love with the apothecary's pretty daughter; but Newton never married and the lady was wed to another.

Newton was nurtured in an atmosphere of political and religious strife during a period in which the English people were driving painfully towards "the ideology of ordered freedom, which has been the greatest contribution of English thought and experience to the civilisation of the human race". The Journals of the House of Commons had long been closed to the public, and it was not until Newton was eleven years old that Parliament proclaimed itself the authentic dispenser of its own news through *Mercurius Politicus*, which received its material at the whim of the Clerk of the House. England had to wait a further forty years for that stroke which freed the Press and printing from legalised official interference except in time of war, and which has been by Macaulay so vividly described. Newton represented as truly as any of his contemporaries the spirit of the new age of free enquiry and expression, and it is no occasion for surprise that we should find him in middle life stoutly confronting the notorious Judge Jeffreys himself, resisting the insidious aggression which the Crown sought to practise in University affairs and consenting to represent his University in the Convention Parliament of 1689. He was a very perfect Christian democrat.

Hundreds of years later we have witnessed the spectacle of Albert Einstein,

next in the honoured line of succession to build an entirely new "System of the Physical World", spurned and rejected by those who had basked in the glory of his greatest discoveries, taking up the cudgels once more in the defence of freedom and acting as an itinerant ambassador of peace and goodwill.

In June, 1661, Newton entered Cambridge, and three years later he was elected Scholar of Trinity College. In 1665 he became B.A., but the place which he reached in the final list is shrouded in mystery, for the order of seniority is provokingly omitted from the Grace Book of that year.

Within a few months he had invented his method of "fluxions" and turned his attention to the investigation of the properties of colours and light. He had obviously assimilated the ideas of his great contemporaries, Descartes and Pascal, in algebra and was probably led to the differential calculus by Fermat's method of drawing tangents. It seems evident that he was preparing a plan of campaign against those outstanding problems of astronomy which had been handed on from Kepler and Galileo and which must have proved a tempting target for so vigorous and original a mind. At all events he began to consider ways and means of improving existing telescopes, experimented with the task of grinding optic glasses to a figure other than spherical, and procuring a triangular prism discovered the "unequal refrangibility" of light.

This last discovery has led to the most spectacular applications in modern science. It has enabled the modern physicist not only to explore the innermost recesses of the atom, but also to extend his experiments to the sun and distant stars, and to investigate the properties of matter under physical conditions which it would be totally impossible to reproduce in any terrestrial laboratory.

As a graduate of one year's standing he had elucidated the complex problem of chromatic aberration and, being thus logically led to abandon his attempts to improve the refracting telescope, he took up the development of the reflecting type, which had been proposed by James Gregory of Aberdeen.

But when the great plague intervened and the University was dispersed he had perforce to abandon his "glass-works", and he retired to the quiet seclusion of Woolsthorpe Manor to meditate upon the two other topics which were to bring him even greater renown, namely, the calculus and celestial mechanics. Of this period he wrote: "In the beginning of the year 1665 I found the method of approximating Series and the Rule for reducing any dignity (power) of any Binomial into such a series. . . . The next year . . . in May . . . I had entrance into the inverse method of Fluxions. And the same year I began to think of gravity extending to the orb of the Moon, and . . . from Kepler's Rule of the periodical times of the Planets. . . . I deduced that the forces which keep the Planets in their orbs must be reciprocally as the squares of their distances from the centres about which they revolve; and thereby compared the force requisite to keep the Moon in her orb with the force of gravity at the surface of the earth, and found them answer pretty nearly. All this was in the two plague years of 1665 and 1666, for in those days I was in the prime of my age for invention, and minded Mathematics and Philosophy more than at any time since."

Twenty years were to elapse before the publication of these ideas, and much has been made of this delay. Some have maintained that he adopted an erroneous estimate of the earth's radius in seeking to verify the theory in relation to the motion of the moon; others that he was unable to integrate the separate attractions at an external point of the multitude of particles which go to make up the massive bulk of the earth, but possibly it was due to his reluctance to publish his results before the complete edifice could be exhibited in all its grandeur.

But the intrinsic value and importance of his work on optics and dynamics alike were becoming recognised, and when he returned to Cambridge in 1667 he was elected a Fellow of his College.

In the following year he constructed with his own hands a reflecting telescope with the object of testing out his law of gravitation on the satellites of Jupiter, and in the following year Isaac Barrow resigned the Lucasian Chair of Mathematics in favour of his brilliant disciple.

Under the Statutes attaching to the Chair, he was required to lecture for about an hour at least once a week during term, and also to conduct tutorials on two days per week during term and one day per week during vacations, if in residence.

He lectured on Optics, Algebra and Mechanics to a select group, and, if no one turned up to his lecture, he returned without complaint to his private experiments and meditations.

At a later date his notes on Algebra were written up by William Whiston, one of his pupils, and published under the title of *Arithmetica Universalis*.

Three years later he was elected a Fellow of the Royal Society.

Never having taken holy orders Newton was afraid lest his Fellowship at Trinity might be withdrawn. This made him anxious about his financial position, and he actually offered to resign from the Royal Society. How different were the circumstances under which Einstein in our day relinquished in exile his membership of the Berlin Academy of Science! The situation in Newton's case was saved by his being allowed to remit the weekly payment of one shilling to the Society, and presently he received a Patent from the Crown allowing the Lucasian Professor to retain his Fellowship without becoming a priest.

This act of royal grace, taken in conjunction with the building of the Royal Observatory at Greenwich about the same time, goes a long way to retrieve the reputation of the "Merry Monarch", who had also chartered the Royal Society.

One happy result of this easing of his financial position was his ability immediately to contribute to the expense of the building of a new library in Trinity College. In later life, when he had left Cambridge and become comparatively well off, he displayed great generosity not only towards his old College but also towards the Royal Society and the parish church at Colsterworth.

During the period under review Newton communicated several valuable results in optics to the Royal Society, but the next twenty years of his life were spent for the most part in quiet study at Cambridge. He was devoid of purely personal ambition and reluctant to publish his work, for he was touchy and sensitive in face of opposition and criticism.

But skilfully coaxed by Edmund Halley, the Secretary of the Royal Society, Newton at last consented to write up for publication his astronomical and dynamical researches. There was probably never a man who concentrated so severely and so rigorously upon the task to which he had set his hand. All material comforts were forsaken in the throes of mathematical composition. At a Council Meeting of the Royal Society held in 1686, the President—none other than the famous diarist, Samuel Pepys—was desired to licence for publication the *Philosophiae Naturalis Principia Mathematica*, and the cost of publication was most generously guaranteed by Edmund Halley. This masterpiece, which has not extravagantly been described as the most original creation of the human spirit ever to be produced, was published in the following year.

No attempt at a full description of the contents is possible here. The First and Second Books built up, on foundations laid by Galileo, the body of doctrine

now known as the "Classical Mechanics". The Third Book cut adrift from the limitations of traditional thought and created a new conception of the "System of the Physical World". Never was so large a mass of natural phenomena brought within so comprehensive, so unified and so elegant a scheme.

The vindication of Newton's theory of universal gravitation was even more spectacular than the subsequent developments of his optical discoveries. A great comet had appeared in 1682 and had been carefully observed by Halley; on referring to records of previous cometary appearances, he suspected that comets which had been noted in the years 1607 and 1531 were really one and the same body following their prescribed courses round the sun and reappearing at nearly regular intervals of seventy-six years. By the application of Newton's treatment of approximate parabolic orbits he looked forward into generations still to come, and predicted that the celestial wanderer would return in 1758 or 1759. Newton passed away in 1727 and Halley was left to maintain a solitary vigil. He too had been dead for seventeen years when the reappearance occurred within a month of the expected time. Halley's comet returned promptly to schedule in 1835 and 1910.

The theory of planetary perturbations led to the discovery of the planet Neptune in the nineteenth and to that of Pluto in the twentieth century. Without the guiding principle of gravitation to assist him, Einstein could not have developed his general principle of relativity.

Newton was more of an originator and a developer than a challenger. He gathered together all the scattered threads of scientific knowledge which were floating idly in the air, and by imparting to them a new strength and direction created a marvellous pattern, which has set the style and fashion for generation after generation of mathematical physicists right down to our own time.

He did not choose to regard himself as a pure mathematician, but rather as a humble searcher after Nature's laws. Yet he discovered the Binomial Theorem and the use of infinite series, invented the Calculus, made most valuable contributions to the theory of curvature, discussed the singularities of algebraic plane curves, developed a numerical method for the solution of algebraic and transcendental equations, and created the interpolation formula which bears his name. His hereditary enemy, Leibnitz, declared that of all the mathematics created up to his time, the better half was due to Newton.

During the years which followed the publication of the *Principia*, Newton's contacts with London life as a member of Parliament rendered him conscious of the limitations and anomalies of his own financial position. He was unable to yield to the generous impulses by which he was animated and, although most indignant at the idea of influential wires being pulled on his behalf behind his back, he eventually accepted the office of Warden and later Master of the Mint. To the English mind there is nothing grotesque in this spectacle of the greatest scientist which this country has ever produced being thus removed from the scene of his greatest triumphs and henceforth suffering the severest restrictions in the exercise of his unique powers. The appointment in 1696 was a bar to the further prosecution of his researches in physical astronomy. His later scientific work was considerable and retained its peculiar quality to the end, but it was carried out by snatches and in the intervals of business.

In 1703 he was elected President of the Royal Society, and re-elected annually for the remainder of his life. In the following year he published his book on Optics, which contained his theory of "fluxions", and which like the *Principia* eventually ran to three editions.

In 1705 he was knighted by Queen Anne at Trinity College, but whether in recognition of his valuable services as Master of the Mint or in acknowledgment of his pre-eminence as a scientist is not recorded. The same year witnessed his failure to secure re-election for a third term to Parliament.

Newton's approach to matters of religion was rational, though devout. He realised the limitations of science and maintained his interest in theology throughout a long life, but he was not a regular attender at the College services while at Cambridge; nor did he degenerate into a religious neurotic as Pascal did in the last years of his life.

Newton was not always amiable in his personal relationships. He was not on very good terms with Robert Hooke, who opposed the corpuscular theory of light and developed the engaging propensity of claiming as his own some of the more important discoveries of the great master. Nor were his personal relations with Flamsteed, the first Astronomer Royal, whom he bombarded with questions about the motion of the moon, always of the best. The famous and long-standing feud between Newton and Leibnitz reflects credit on neither of these great men. Each invented the calculus independently of the other, and the ridiculous squabble about priority has become a matter of small account now that the "relativity of simultaneity" has been established.

The progress of physical science depends upon the interplay of experiment and observation on the one hand and the mathematical analysis of the results obtained on the other, both activities being carried out at a high level of competence. New observations lead to new analysis, and this in turn suggests fresh experiments. Newton occupied a unique position inasmuch as he combined both qualities in the one personality.

The subsequent history of scientific thought has shown that practical achievement ever lags behind imaginative conception, and the mathematicians have never failed to anticipate the needs of the physicists by providing new weapons of attack upon the mysteries of nature.

In the Golden Age which followed upon the Newtonian era it gradually came to be recognised that, in spite of Newton's famous dictum, "non fingo hypotheses", universal gravitation, with its implication of a perfect "aether" pervading the whole of a Euclidean space, remained a pure hypothesis, and that the ideas of absolute distance, time, velocity and all the rest, which had been accepted as axiomatic by Newton, were little more than figments of a disordered imagination. It is perhaps something of a paradox that this change of outlook should have been induced by the modern development of electrodynamics and optics—the fruition of seeds implanted by Newton.

One of the outstanding pioneers of this new age was Nikolas Ivanovitch Lobatchewsky who, having the audacity to challenge the validity of the geometry of Euclid and Pythagoras as applied to the physical universe, has not inaptly been described as the Copernicus of geometry. It took over 2000 years to emancipate the mind of man from the belief that Euclidean geometry represented *absolute* truth in its purest form, and it was this Russian professor in the University of Kazan who did it. Thus consider the famous Theorem of Pythagoras and suppose that, instead of drawing our triangle on the idealised plane imagined by Euclid, we draw it, as Archimedes was wont to do, upon the sandy surface of the earth, idealised to the extent of being supposed a "perfect" sphere. Since we cannot draw a straight line to lie within the surface of the sphere, we must replace straight lines by "geodesics" in the surface, *i.e.* lines of shortest (or longest) length joining any two points. The problem is now reduced to one in spherical trigonometry, and it is well known that, if $BC^2 = AB^2 + AC^2$, then either the angle at A is *not* a right angle or one of the sides at least is *not* "straight".

It is important to realise just how this break-away from ancient geometry

has been attained. The Euclidean plane has been distorted and endowed with the property of " curvature ". The sphere is obviously a surface of constant (total) curvature ; but it is equally easy to visualise a surface for which the curvature may vary not only with position but also with time. The smoothly rippling surface of a lake is an example of a non-Euclidean " space " possessing this property.

The appropriate parametric representation of such surfaces and the treatment of their curvature was the work of the celebrated Gauss, who was led by mainly practical considerations to investigate the deeper properties of surfaces, for he acted as scientific adviser to the Hanoverian and Danish governments in an extensive geodetic survey carried out by them during the years 1821–1848.

The non-Euclidean two-dimensional " space " which we have considered finds itself immersed in a three-dimensional " manifold " which is itself Euclidean, and we may employ the Theorem of Pythagoras in this manifold to obtain the conventional formula for an element of arc on the sphere, namely,

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2 \theta \, d\phi^2.$$

It was left to Riemann entirely to dispense with this leg-up from Euclid and to create on a precisely similar pattern a non-Euclidean " space " of any number of dimensions, defined by the differential quadratic form :

$$ds^2 = g_{\mu\nu} d\theta_\mu d\theta_\nu.$$

The g's, which are neither more nor less mysterious than the dots which once puzzled a famous Chancellor of the Exchequer, depend upon the parameters θ. For each particular system of g's a corresponding type of space is defined and, if it should happen in three dimensions that $g_{\mu\mu} = 1$, and $g_{\mu\nu} = 0$, our old friend Euclid emerges as a particular case. Riemann went on to develop the theory and to generalise the idea of Gaussian curvature for such spaces.

While all this progress was being made in the mathematics of the nineteenth century, the physicists in their laboratories had been encountering complexities, which led to a re-examination of the fundamental bases of their beliefs. It gradually came to be understood, for example, that unless size were to be regarded as some metaphysical property of a material body, and not simply the result of measurement, then even the distance between two points of a rigid body was a relative conception and depended upon the motion of the particular observer who carried out the measurement. This debunking process was completed by Albert Einstein, who challenged the axiom that " two events can occur in different places at the same time ".

The resulting confusion was for a time considerable, until Minkowski and others came to the rescue by suggesting that perhaps the complexity was due to a too bigoted adherence to the geometry of Euclid and might be overcome by regarding the world of physical phenomena as a four-dimensional manifold in the space-time sense.

Einstein took up this idea and, by an ingenious selection of one of the infinite variety of four-dimensional geometries of Riemann, succeeded in creating his famous relativity theory of gravitation, which was first published in its complete form during the last war in November, 1915, when he was still a professor at the Prussian Academy of Science in Berlin.

This theory represents the space of everyday life rather as the three-dimensonal " surface " of a rippling lake, finite but unbounded, the ripples being matter and their movement representing the large-scale motions of the physical universe.

It is characterised by the following differential quadratic form in four dimensions :

$$ds^2 = (1 - 2m/r)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 - (1 - 2m/r) dt^2.$$

Here ds denotes not merely an element of distance but a generalised "interval" involving both distance and time, and m is the mass of the attracting particle in gravitational units.

Newton's formulation of the laws of motion for a particle in a gravitational field is now replaced by the single statement : "Every particle and light-pulse moves so that the integral of ds between two points of its track is stationary."

The theory has been subjected to three celebrated tests. Two are observational and cover speeds both of low and of high order. The third is experimental and very delicate. They are as follows :

(1) The famous large discordance in the observed position of the perihelion of the planet Mercury (8″ of angle per century) has been explained in an entirely natural manner.

(2) The deflection of a ray of light grazing the limb of the sun has been confirmed as 1″·75 as against 0″·87 or zero on previous theories.

(3) Physicists are generally agreed that the period of an atom vibrating in the photosphere of the sun appears longer than that for an identical atom in a terrestrial laboratory.

No one will pretend that the last word has been said. After the National Socialist revolution in Germany, Einstein, like many refugees before him, became an emigrant to the United States of America, and gratefully accepted sanctuary within the portals of Princeton University. The threat of a breakdown in civilisation itself, which seemed almost imminent in 1939, is now being averted, and no disaster of lesser magnitude would suffice to arrest the progress of mathematical research and discovery. T. A. B.

QUEENSLAND BRANCH.

Report for the Year 1942–3.

The Annual Meeting was held on 22nd May, 1942 ; the Annual Report and the Statement of Receipts and Expenses were presented and were adopted, after which the officers for the coming year were elected. The subject of the Presidential Address by Professor Simonds was "The Beginnings of Mathematics".

During the year two General Meetings were held : at the first of these, held on 28th August, Mr. E. W. Jones read a paper on "Computations", and at the second, on 30th October, Mr. J. P. McCarthy read a paper on "The Wallace (or Simson) line and the Wallace point".

The Statement of Receipts and Expenses shows a credit balance of £10 18s. 9d. The attendance at meetings has been affected by the fact that certain members are on duty with the Forces. The number of members is 26, of whom 8 are members of the Mathematical Association. In spite of acute difficulties, copies of the *Mathematical Gazette* reach us in due course and are circulated amongst Associate Members.

The Committee is as follows : *President*, Professor E. F. Simonds ; *Vice-Presidents*, Messrs. S. Stephenson and I. Waddle ; *Hon. Secretary and Treasurer*, Mr. J. P. McCarthy ; *Members*, Miss E. H. Raybould, Messrs. R. A. Kerr, E. W. Jones, J. C. Deeney, P. B. McGovern.

J. P. McCarthy, *Hon. Secretary*.

REFLECTIONS ON THE TEACHING OF MATHEMATICS.

By S. Weikersheimer.

1. *Introduction.*

At the present time, when the reform of the root-principles of education is being considered, it may be useful to study systems, organisations, didactic and methodic experiences of other countries, and compare them with the proposals recently made by various committees and individuals, proposals that might become the basis of educational regulations of the near future. The comparison of the study and teaching of mathematics is the more justified since they will be certainly less affected by differences in national temperament and conditions than the study and teaching of any other subject.* This has been one of the purposes of the meeting of school teachers from occupied countries working in British schools, in conference with British teachers, in August 1943. These intensely practical studies of comparative education are invaluable at this time.†

The object of this article is to give an account of the teaching of mathematics in secondary schools of Bavaria from 1914 till 1933, for comparison with existing regulations in this country, and with the proposals of committees, especially those of the Norwood Committee. Why just Bavaria, and why from 1914 to 1933? The answer to the first question is in part personal. I attended a Bavarian secondary school, studied mathematics and physics at Bavarian universities, and was a master at Bavarian secondary schools for twenty years. Further, Bavaria was the first State on the Continent where the proposals ‡ of Felix Klein on the reform of mathematical instruction, made at the Congress of the German Natural Scientists and Physicians at Merano in 1905, were officially put into effect, in 1907 § only at the Oberrealschulen—the natural science-mathematical type of secondary school—which were newly founded in Bavaria at that time, and at all Bavarian secondary schools in 1914 § when new teaching regulations for all secondary schools came into force. This is the reason for the date 1914. The other limit must be 1933, as the instruction meted out under the Nazi government aimed at a very different goal.

2. *Administration and Organisation.*

The secondary schools in Bavaria (similar to those in other German States: Saxony, Württemberg, Baden) were nearly all State schools. The Minister of Education was responsible to the State Parliament. The headmaster and the permanent masters were civil servants. The headmaster was responsible to the Minister of Education.

There were three types of secondary schools : (1) the Gymnasium stressing classics, (2) the Realgymnasium with modern languages (mostly English and French), (3) the Oberrealschule emphasising mathematics and natural science. Each secondary school had nine forms, the pupils aged 9 to 17, or 10 to 18. When the pupils entered the secondary school they had to be 9 years ; this was later altered to 10. The boys and girls had to do all the subjects taught at the school concerned, and were examined in all the main subjects fixed by the Ministry. Mathematics was one of these subjects for all schools.

* J. W. A. Young, *TMP*, p. 125 and p. 135.

† *The Times, Educational Supplement*, p. 421.

‡ See p. 15. § See footnote (†) on p. 15.

3. *Curricula.*

The curriculum was uniform for each type of secondary school throughout the State. The curricula were issued for all subjects together with statements on the aims of the teaching and with methodic remarks. During the three terms of a school-year the pupils had to write a number of tests in each subject as prescribed in the Ministry regulations. The tests were to be corrected, marked (1, 1–2, 2, 2–3, 3, 3–4, 4 ; later altered into a scale of 5 marks), and submitted to the headmaster. At big schools the headmaster shared the inspection with one or two second masters. The marks on the tests and for oral contributions of a pupil in class formed the basis for judgment at the end of the term, and especially at the end of the school-year when the advance of each pupil was decided at a meeting of the entire staff of the school (of course mainly by the pupil's masters). If a boy was not allowed to advance to the next form he had to attend the same form again in the following school-year. Under certain circumstances (*e.g.* if he would have to take the same form for a third time) he was compelled to leave the secondary school, and to go back to the elementary school, if under 14. The decision rested with the majority of the staff. The reports at the end of each school-year contained a judgment on the behaviour, diligence, character, and other particulars of the pupil, besides the marks in the subjects, expressed in numbers or in equivalent words (very good, good, —later fair was inserted here—, satisfactory, unsatisfactory). The boys were not divided into graded forms according to their knowledge ; on the contrary, the headmaster endeavoured to distribute the gifted and the less capable boys uniformly, especially those who had to attend a form for the second time. I cannot agree with the system obtaining in this country by which the boys with the marks—say—300 to 240 are put in set (or division) B1, 240 to 180 in set B2, and so on. There is no spirit in B5, and if there is a boy who is not quite so weak, he generally feels bored and lowers his own standard to that of his set. To the teacher, such a set is a torment. Another disadvantage is the unrest in the school building between two periods, and the loss of time, even if this is only a few minutes. I think a compromise would be quite satisfactory : collect all really able boys in one form, but divide all others uniformly, which would leave them in their forms.

4. *Examinations.*

A complete secondary school had nine forms. There were quite a number of secondary schools with six forms only ; at these schools the boys had to take an examination which approximated to the school certificate examination, or rather the new examination of the Norwood Report. It was actually an internal examination ; at complete secondary schools it was dropped. The examination after nine years can be compared with the higher school certificate examination. The subjects of the written examination (all subjects were compulsory) at a natural science-mathematical school were : divinity (with exceptions), three languages, mathematics, physics, chemistry with biology, arts. Boys who did not do well in a subject had to take an oral examination, at least in this subject. The level of requirements was not so high as it is in the higher school certificate examination in this country ; but it should be realised that pupils could not choose a small number of subjects, for instance mathematics, physics, and chemistry only, not even at such a mathematics and science school. Mathematics was a subject of examination at all schools and in all examinations. The mathematics papers for Gymnasiums and Realgymnasiums were set by the Ministry ; at the Oberrealschulen the masters teaching the upper forms had to submit three (sometimes two) proposals to the Ministry who chose one of them. All scripts were corrected and marked by the teachers of the upper forms concerned ; the judgments were to be

examined by a second reporter. In general, I can say from my experience that the level of requirements is higher where the examinations are centralised, although where they are not, it may sometimes be found that a teacher may set a higher standard for a special part of his subject he has dealt with at length. But this is—in my opinion—not the purpose of an examination; rather it should show that the pupil masters a certain stock of knowledge in the subject which will enable him to continue his study at the university or in industry. Therefore I do not see any advantage in the proposal to alter the present system of the examinations in this country. This is certainly the best I have known in my practice extending over thirty years. It compels teacher and pupil to work hard, and it is really impartial. Alterations of detail might be useful, *e.g.* the setting of papers, correcting and marking of scripts could be more or less transferred to experienced masters at high schools, or to inspectors.

5. *Teachers.*

On the Continent the mathematics teachers had generally to study at the university for four years. In Bavaria they had to combine it with the study of physics; in Baden and Württemberg the combination was mathematics-physics-chemistry. Then they had to attend a training college for one or two years. This time was well spent, the candidates learned in a short period what they would have otherwise acquired by experience in over many years. The discussion of didactic, methodic, and pedagogic problems was continued at regular or occasional meetings of all mathematicians at the big schools.

It was a regulation that the headmaster had to inspect every teacher's lessons from time to time. Every two years he had to report about the teacher's qualification to the Ministry, actually to the ministerial official dealing with mathematics. Sometimes such an official inspected the lessons of all teachers at a school. After the inspection there was generally a meeting of the masters who taught the subjects of his province; every master could express his opinion quite candidly, and there was no stiffness among the experienced teachers, and the inspection was not considered disagreeable. The training college offered a good preparation for this, because the inspection was carried out there also in a similar way.

A mathematician, actually every teacher, should be enthusiastic and devoted to his profession. For this purpose he must be free from financial worries. A salary adequate to his education, his ability, and his responsibility will secure him the social position he deserves. I must say the financial and social position of a master at a secondary school in most of the continental countries stood on a much higher level than they do in this country. It is understandable that young people (and men and women in the forces who do not know what to do after the war) are not inclined to devote themselves to the teaching profession.*

6. *Aims of the Teaching of Mathematics.*

Mathematics in secondary schools is to be taught not only for the sake of logical training and its educational importance, but also for the creation of a foundation for the understanding of the natural sciences and to enable us to view mathematically the world of phenomena surrounding us. This last purpose was especially stressed by the reform movement starting about 1900. This made the teaching aim more or less utilitarian. The seventeenth century, the Golden Age, shows how mathematics benefited when natural science threw up problems which could not be solved with the mathematical equipment of

* See *The Times, Educational Supplement,* 1943, pp. 403 and 392; also Young, J. W. A., *TM*, p. 408.

the time, and new branches of mathematical thought had to be born. Thus a cross-fertilisation took place, to the mutual advantage of both sciences. In the eighteenth century mathematical teaching gave prominence to the utilitarian aims, whilst the nineteenth century emphasised the logical training as the main purpose. These two aims of instruction in mathematics should be combined. The unifying idea in the different branches of mathematics is the thinking in terms of the function. It was the merit of the three great mathematical reformers, J. Perry in England, H. E. Moore in America, and Felix Klein * in Germany, who devoted themselves with all their power and influence to the problems of mathematics teaching in suggesting changes in the selection of subjects to be dealt with, and clearer methods of instruction. The outstanding figures in France were J. Tannery and E. Borel. The Congress of German Natural Scientists and Physicians at Merano in 1905 accepted all Klein's proposals except that for the teaching of calculus in the upper forms of the Gymnasium ; this proposal was only recommended.†

The instructions for the teaching of mathematics in the official Bavarian curriculum were as follows : The mathematical instruction should aim at sureness and skilfulness in calculation, knowledge of the most important theorems, of the methods and applications of elementary mathematics, and at the introduction to the calculus and analytical geometry of the conic sections ; this last was to be confined to the elements at the Gymnasium and Realgymnasium. In all forms an adequate degree of skill was to be reached in translating problems into equations, and in working out exercises. Apart from this task the purpose of mathematical teaching was the strengthening of the power of space intuition, the training in habits of clear and logical thought, especially of the concept of a function.

The methodic regulations contained the following remarks : The explanations were to be simple and lucid, the memorising to be reduced to a minimum, peculiarities requiring special artifice were to be avoided.

As for the teaching of *arithmetic*, I want to emphasise a few of the instructions : problems to be practical ; the pupils to become familiar with the objects of modern life of trade and commerce ; estimates of results of problems to be made frequently ; problems to be solved sometimes by means of equations ; plotting of graphs.

Algebra. For training in the use of the function concept, equations to be solved graphically ; for the better understanding of algebraic laws, letters frequently to be replaced by numbers ; complicated calculations were to be avoided ; the early use of logarithms, and of the slide rule from the sixth form (see later).

Geometry. First, practical examples (surveying of the schoolroom, schoolyard), exercises with compasses and set squares, then systematical treatment, simple proofs and fundamental constructions. Diagrams on the board and in the note-books to be drawn carefully. Solve problems of construction, which can be done without special artifice, the discussions to be complete.

Trigonometry. Occasional exercises of surveying, the trigonometrical functions, first considered as ratios, then as distances in the unit circle, graphs of trigonometrical functions.

* Young, J. W. A., *TM*, p. 4, says about Felix Klein that he has no doubt exercised a stronger personal influence on the development of American mathematics and mathematicians of the present day than any other European.

† With regard to page 12, Young is therefore not right when he writes in his *TM*, p. 182, that "this report has been too recently published to have yet affected the curriculum in actual use, but that it will have decided influence in shaping curricula of the near future cannot be doubted".

Solid Geometry. The pupils to make models of simple mathematical solids, drawings of these bodies in orthogonal and oblique projection. Cavalieri's principle.

Analytical Geometry. The most important properties of conic sections. At the Oberrealschulen, the treatment of the projective geometry was prescribed, but it was dropped a few years later.

Calculus. The basic ideas and the simplest rules of calculation. As a general rule, all mathematics lessons were in a form to be taken by one teacher.

The mathematics syllabus at the natural science-mathematical type of secondary school :

1st Form (lowest form, age 9–10, 4 lessons a week).

The four fundamental operations with whole numbers, applications to simple problems, measurements, weights, coins ; rules for the divisibility of numbers by one-figured numbers (except 7), factorisation into prime factors. Mental arithmetic. Occasionally graphs.

2nd Form (4 lessons a week).

H.C.F. and L.C.M., formation of ordinary and decimal fractions. Fundamental operations with ordinary and decimal fractions, conversion of ordinary fractions into decimal fractions, and finite decimal fractions into ordinary fractions. Practical examples. Simple examples of proportion. Occasionally graphs.

3rd Form (5 lessons a week).

Arithmetic. Short calculation with decimal fractions (simple examples), compound proportion. Calculations of percentages, interest, discount. Occasionally graphs.

Geometry. Solids and planes, plane figures considered as portions of plane-surfaced solids, then by themselves. Explanation of direction, angle, parallelism, symmetry. Occasionally surveying. Then systematical treatment of angles and triangles. Construction of triangles from given elements. Theorems of congruence ; right-angled, isosceles, equilateral triangles. Loci.

4th Form (5 lessons a week).

Arithmetic. Problems from daily life, physics, geometry, and solid geometry (using short computation), and application of linear equations as preparation to the systematical treatment in algebra.

Algebra. The four fundamental operations with letters. Positive and negative numbers. Development of the function concept by the discussion of the dependence of the value of an expression on other quantities. Numerical and graphical solution of linear equations with one unknown. Application to practical examples, notably problems of mixture and division. Equations with proportions as far as necessary in geometry.

Geometry. Parallelogram and trapezium. Simple problems of construction of triangles and quadrilaterals. Circle. Figures equal in area, mensuration. Conversion and dividing of figures.

5th Form (5 lessons a week).

Algebra. Numerical and occasionally graphical solution of two linear equations with two unknowns. Powers and roots with integral indices. The square root of a number. The graph of a function of the second degree. Numerical and occasionally graphical solution of quadratic equations with one unknown and simple cases of two equations with two unknowns. Applications.

Geometry. Ratios of distances. Similarity. Regular polygons. Mensuration of circle. Problems of construction, especially algebraic-geometrical ones.

Solid Geometry. The oblique projection of square and rectangle, cube and prism as silhouettes of parallel rays of light. Theorems on the position of straight lines and planes in the space. The most important properties of prism and pyramid. The oblique and orthogonal projection of these solids in simple positions. Surface and volume of these bodies.

6TH FORM (5 lessons a week).

Algebra. The graph of the exponential function. Logarithms. Explanation and occasional use of a slide rule. Arithmetical and geometrical progression. Application to calculations of compound interest and amortisation (simple examples from daily life).

Solid Geometry. The most important properties of cylinder, cone, truncated pyramid and cone, sphere ; surface and volume of these bodies. The regular polyhedra, especially tetrahedron, cube, octahedron.

Trigonometry. Introduction of the trigonometrical functions ; graphs of these functions in the first and second quadrant. The fundamental formulae of goniometry. Simple problems on triangles with applications.

UPPER FORMS : 7TH FORM (5 lessons a week).

Algebra. The complex numbers, their representation in the Argand diagram, the fundamental operations with complex numbers, including powers with real indices (numerically and graphically). De Moivre's theorem, application to the solution of equations of the form $x^n - 1 = 0$. Formation of a polynomial in x out of linear factors and factorisation of the polynomial, with a view to the theory of equations of high degree. Approximate solution of equations by means of the *regula falsi*. Simple examples of the graphical solution of equations.

Trigonometry. Generalisation of the trigonometrical functions of any angles and completion of their graphs. Short revision of the most important goniometrical formulae and of the theorems on the general triangle.

Spherical Trigonometry. Deduction of the fundamental formulae by means of the spherical pyramid. The right-angled spherical triangle and the simple cases of the general triangle. Area of the spherical triangle. Applications to simple problems of solid geometry, and astronomy with consideration of the following subjects : coordinating systems of the celestial globe, rise and set of stars. Determination of time.

Descriptive Geometry. Systematical compilation of the methods (gained in solid geometry) of descriptive geometry. Elementary problems on point, straight line, and plane. Intersection of plane-surfaced solids with straight lines and planes. Simple problems of intersection of such solids.

8TH FORM (5 lessons a week).

Analysis and Calculus. The simplest theorems of combinations, permutations, and distributions as a preparation to the binomial theorem. The binomial theorem for positive integral indices. Relations between roots and coefficients of an equation. The limit concept. Explanation by various examples. Introduction of the differential coefficient and its deduction for a power, $\sin x$, and $\cos x$. Differentiation of a sum, difference, product, and quotient. Application : tangent to a curve, solution of maxima and minima problems, and problems of mechanics and physics.

Analytical and Projective Geometry. Systematic revision of results of the determination of a point, and of the graphs done before. The straight line and the pencil of straight lines. The circle and systems of circles. Tangent, pole, polar to a circle. Harmonic properties. Harmonic position of four points on a straight line, and four straight lines through a point. The cross-

ratio of points on a straight line, and of rays in a pencil, projective properties of cross-ratio.

Descriptive Geometry. Orthogonal projections of a circle. Representation of cylinder, cone, and sphere. Intersection of these bodies with straight lines and planes. Introduction to the theory of perspective drawing.

9TH FORM (5 lessons a week).

Analysis and Calculus. The first theorems of the theory of series. Criterion of convergence from the comparison with geometrical progressions. Elementary deduction of the series of e^x, $\sin x$, $\cos x$, $\log(1+x)$. $\log x$ and e^x with their differential coefficients. The second differential coefficient in Geometry (curvature) and Physics (acceleration). The conception of the indefinite integral. The method of substitution, integration by parts, and the method of partial fractions. The definite integral. Simple calculations of areas, centres of gravity, and volumes.

Analytical and Projective Geometry. The curve of second order as a plane section of a vertical circular cone (Quetelet-Dandelin's theorem). Deduction of the equations of ellipse, parabola, and hyperbola from the properties of the foci. Tangent, pole, polar. Generation of the conic sections by congruent pencils of rays. Pascal's theorem. Problems of construction by means of Pascal's theorem. Problems on loci representing conic sections.

Descriptive Geometry. Continuation of the exercises of descriptive geometry. Perspective representation of simple bodies by orthogonal and oblique projection.

Projective geometry proved too hard for the average boy, so after a few years it was dropped as a subject to be treated and transferred to the university. The syllabus for the Gymnasium and Realgymnasium was not so comprehensive as that of the Oberrealschule, but it included the elements of analytical geometry with problems on circles and conic sections, and the introduction to calculus up to the differentiation of $\sin x$. Apart from this, the right-angled spherical triangle was to be dealt with, and its application to astronomy, the elements of which were to be taught. Descriptive geometry was not treated, neither the theory of complex numbers and relations between roots and coefficients of terms in an algebraic equation of a higher degree.

7. *Methods of Instruction*.

The methods of instruction will always differ because they depend upon the individual teaching ability of the masters. But on the whole there are some essential differences. A mathematics lesson in Bavaria generally was an hour of heuristic work, a continuous discussion between the teacher and the Form (only one boy was to speak at a time) under the guidance of the master. This was a root-principle taught by the masters of the training colleges to the intending teachers, and it was carried out in practice, at least by conscientious mathematicians. It did not happen that the whole Form was occupied in doing exercises throughout the lesson. New topics at any rate had to be treated in class. The homework consisted of a number of problems of the same type, nothing new in the way of principle. Another point: the results in geometry tests and examinations are generally not so good as those in algebra. What is the reason? I think that proofs of theorems are too hard for most of the boys. Therefore in Bavaria a much greater stress was laid on problems of construction (of triangles or other figures) than is usually done in this country. Furthermore, arithmetic was not taught beyond the Bavarian fourth Form and was no subject of examination. The revision of the matter in a subject treated in previous Forms was much more briefly dealt with—if at all—than is generally the case in this country.

The mathematics textbooks were mostly used as collections of exercises only; it happened very seldom that a boy learned how to solve algebra problems, or revised theorems or proofs from his textbook: rather he wrote them carefully and neatly in his note-book as they were treated in class and learnt the subjects from it. In geometry the figures had to be drawn properly and accurately, using ruler, set square and compasses. The characteristic features of the mathematics textbooks published in this country—as far as I have used them at secondary schools, or as far as I have been able to study them—are: the ample selection of riders, the good and comprehensive choice of theorems and their proofs, the great number of examples, in particular of practical examples, and the lucid text of explanation. The standard of these books is definitely much higher than that of the corresponding textbooks used in Bavaria or in Germany.

8. *Conclusion.*

I know very well that this discussion is not exhaustive, but it will give an idea of the solution of problems which will arise when the new proposals such as the creation of the three types of secondary schools become accomplished facts. J. W. A. Young * says: These are " problems with which we are now grappling; it would be foolish to refuse to consider solutions which the Germans have worked out for the same problems ". Surely, consideration of work done in Germany should not be limited to the field of pure scientific research to the exclusion of German pedagogic experience.

King Edward's School, Birmingham. S. W.

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* J. W. A. Young, *TMP*, p. 135.

GLEANINGS FAR AND NEAR.

1442. In this his greatest triumph [trans-Atlantic radio communication] Marconi got all the credit of doing the " impossible ". At the very time that the transmitter was crackling away at Poldhu and a jubilant Marconi was watching his new receiver at work in Newfoundland, mathematicians were busy proving that wireless reception over distances of more than a few hundred miles was impossible. It was not possible, they said, for wireless waves to scatter sufficiently to get round the curvature of the earth for any greater distance.—A. W. Haslett, *Radio round the World*, p. 28. [Per Mr. F. W. Kellaway.]

CUBE ROOT.

BY H. W. RICHMOND.

1. There is a curiously simple and compact formula *

$$\frac{a-b}{a+b}=\frac{1}{n}\frac{a^n-R}{a^n+R} \qquad \text{(i)}$$

which may be used to derive from a, a first approximation to the nth root of a number R, a second approximation b, very much closer to the true value. Applications of this formula when n has the value 3 are the purpose of this note.

To a reader of the *Gazette* the formula is possibly not new. It is not to be found, so far as I can ascertain, in any treatise or textbook, and has remained unknown; but in 1929 the *Gazette* (Vol. XIV., p. 308) printed in a footnote a statement made by **Reuben Burrow** in the preface to his *Theory of Gunnery* (1779) that a rule for extracting cube root (equivalent to (ii) below) was "far more exact and expeditious" than the common method. Three years later this "gleaning" inspired Mr. **G. W. Ward** to send two contributions to Vol. XVII. (pp. 52 and 127). In the first he showed that the order of the error in b was the cube of that in a; and in the second, by an improvement upon Newton's method of approximation, he obtained the explicit value of b given by (i) for any value of n, anticipating the present paper by ten years. I have to thank Mr. T. A. A. Broadbent for drawing my attention to these facts and dates.

2. Let r denote the cube root of R, n now being 3. By (i),

$$\frac{a-b}{a}=\frac{a^3-R}{2a^3+R}=\frac{a^3-r^3}{2a^3+r^3} \qquad \text{(ii)}$$

and

$$(b-r)=(a-r)^3(a+r)/(2a^3+r^3). \qquad \text{(iii)}$$

We must estimate roughly the errors $a-r$, $b-r$ in a and b, or preferably the *proportional* errors $(a-r)/r$ and $(b-r)/r$, denoted by A and B. (To the degree of accuracy required it is immaterial whether r or a or b is the denominator of A or B.)

B is expressed in terms of A by (iii).

$$B=(b-r)/r=A^3\times r^2(a+r)/(2a^3+r^3)=\tfrac{2}{3}KA^3, \qquad \text{(iv)}$$

where, since a and r are nearly equal, K is nearly 1; actually K lies between a^3/R and R/a^3. The value of A is a little uncertain because r is not known exactly; but since

$$a=r(1+A), \quad (a^3-R)/R=3A \text{ nearly,}$$

the terms $3A^2+A^3$ being neglected.

Numerical examples will now explain the processes.

3. *The cube root of* 6.

R being 6, take as a first approximation $a=9/5$, so that

$$(a^3-R)/R=-21/750, \text{ and } A=-7/750 \text{ roughly.}$$

* Obtained as a particular case of a formula applicable, like Newton's, to equations not necessarily algebraic.

The error in a being less than one-hundredth of a, the error in b will be appreciably less than one-millionth part of b, in fact about two-thirds of this. By (i),

$$\frac{9-5b}{9+5b}=\frac{1}{3}\,\frac{9^3-6\times 5^3}{9^3+6\times 5^3}=\frac{-7}{1479},$$

$$\frac{9-5b}{18}=\frac{-7}{1472}\quad(1472=8\times 8\times 23),$$

$$9-5b=-63/736=-0{\cdot}0855978261\ \ldots,$$

$$b=1{\cdot}8171195652\ \ldots .$$

The actual value is $1{\cdot}81712060\ \ldots$,

and the difference 11 in the seventh decimal place tallies with the forecast.

4. *The cube root of* 1943.

This number is midway between the cubes of 12 and 13 ; 25/2 is a reasonable value for a.

$$\frac{25-2b}{25+2b}=\frac{1}{3}\,\frac{15625-8\times 1943}{15625+8\times 1943}=\frac{27}{31169},$$

$$\frac{25-2b}{50}=\frac{27}{31196}.$$

Hence by division $b=12{\cdot}47836261$.

The tables give $12{\cdot}4783625$; the difference again tallies.

5. It is clear that the method is a powerful one ; moreover, all the operations can be performed on a calculating machine. The cube root of 6 was found to six decimal places from the fraction 6597/3680. Taking this fraction for a we should find for b a fraction giving the root to some 18 decimal places, but the denominator would be a vast number, and such accuracy is beyond any practical need. The value we have found will suggest a better first approximation, 20/11 for example. Or we may employ a formula such as

$$\frac{a-b}{a}=\frac{a^3-R}{pa^3+qR},\quad p+q=3,$$

in which it will be found that B is of the order A^2.

The method can be applied with equal success to square root. A fuller general discussion of these approximation formulae has been submitted to the London Mathematical Society. H. W. RICHMOND.

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should whenever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. If, however, the questions are taken from the papers in Mathematics set to Science candidates, these should be given in full. The names of those sending the questions will not be published.

Applicants are requested to return all solutions to the Secretary.

LINEAR EQUATIONS IN INTEGERS.

BY VIVIAN E. GUMBRILL AND CEDRIC A. B. SMITH.

1. *Introduction.*

The process normally given in books for the solution of linear equations in integer unknowns is dependent on the theory of continued fractions. There is however no need for any such special theory, and the continued fraction process is not the quickest possible.*

But a knowledge of a simpler process seems to be far from general, and so it would seem to be of value to give an account of one such process here, and to see how it may be given a simple arrangement. In addition, as we will see, this process has the good point that it may be used without difficulty to give the general solution of equations with more than two unknowns, and of systems of such equations.

2. *Equations in two unknowns.*

As a specially simple example of the working of the process, let us take the linear equation in two integer unknowns : that is to say, one of the form

$$ax + by + c = 0 \quad \text{.....................................(1)}$$

where a, b, c, x, and y are integers.

Now it may be seen without difficulty that if a and b have no factors in common (as is generally true in practice) then the general solution of this equation is

$$x = x_0 + br \;; \quad y = y_0 - ar$$

where (x_0, y_0) is any one solution, and r is an integer variable. So, it is most frequently enough simply to get one solution of the equation.

Now such a solution may be got by a chain of operations on the given equation (1), each operation giving an equation simpler in form than the one before, so that in the end we come to an equation so simple that its solution may be done at sight.

The first step in the solution is made up of :

(i) the division of the given equation (1) by that one of its coefficients a, b, which has the smaller modulus—let us say the coefficient a. So if $b =$ (say) $ab_1 + b_2$, b_1 being the coefficient in the division, and $c = ac_1 + c_2$, then we may put equation (1) into the form

$$a(x + b_1 y + c_1) + b_2 y + c_2 = 0 \text{.................................(2)}$$

By taking ab_1 to be that multiple of a which is nearest in value to b, we may make $|b_2| \leqslant \frac{1}{2}|a|$, and in the same way $|c_2| \leqslant \frac{1}{2}|a|$.

(ii) We now put in place of the unknown x (the one which has the smaller coefficient a) the new unknown w equal to $(x + b_1 y + c_1)$. In this way we get in place of (1) the new and simpler equation

$$aw + b_2 y + c_2 = 0. \quad \text{...................................(3)}$$

The first step is now complete.

Now it is clear that if x, y are integers forming a solution of (1), then w, y are integers forming a solution of (3). On the other hand, if w, y are integers forming a solution of (3) then, because $x = w - b_1 y - c_1$, x and y will be integers forming a solution of (1). So in this sense (1) and (3) are equivalent.

But clearly we may do the same operation again on (3), and so on, till we

* D. H. Lehmer gives a process very like ours, but using determinants. See "A note on the linear Diophantine equation", *Amer. Math. Mon.*, vol. 48 (1941), pp. 240–246.

come to an equation whose solution may be got without trouble. Working back from this, step by step, we get the solution of (1).

For example if we take the equation,

$$7x+11y+5=0, \quad \text{.................................(4)}$$

on division by 7 we get ($11-14=-3$, etc.)

$$7w-3y-2=0, \quad \text{.................................(5)}$$

on division by -3

$$w-3v+1=0,$$

of which one solution clearly is $v=0$, $w=-1$. Then from (5) $y=-3$, and from (4) $x=4$. This gives one solution, and the general solution is

$$x=4+11r\ ; \quad y=-3-7r.$$

3. *A general solution of linear equations.*

This same process may equally well be used for the solution of a system of equations in any number of unknowns. In order to see how this may be done, it will be a help first to give an arrangement of the solution of linear equations (in which the unknowns are not limited to integer values) such that the values of all the unknowns are given together at the same time by the process.*

The process when the unknowns are limited to integer values will be seen to be very like this.

Let us take as an example the equations

$$\begin{array}{l} \left.\begin{array}{l} x+2y+3z-10=0 \\ 2x+\ \ y+\ \ z-\ \ 9=0 \end{array}\right\} \quad \text{.................................(6)} \\ \hline x \\ \quad y \\ \qquad z \end{array}$$

The two given equations (6) are put into the form in which their right-hand sides are 0. Under them we then put a line, and under the line the three unknowns whose values we have to get.

Now the value of any row in this arrangement will not be changed by the addition of any multiple of the L.H.S. of the top row, for that is 0. So by the addition of the right multiples of the top row, we may make x have a zero coefficient in every row but the top. Here we do the addition of -2 (top row) to the second row, and -1 (top row) to the third. The top row may now be dropped as it is no longer needed, and we get the new arrangement, in which the variable x does not come into the left-hand sides at all.

$$\begin{array}{l} -3y-5z+11=0 \\ \hline -2y-3z+10=x \\ \quad y \qquad\qquad\ \ =y \\ \qquad\quad z \qquad\ =z \end{array}$$

Doing the same process again, by the addition of -2 (top row)/3 to the second and (top row)/3 to the third row, and dropping the top row, we get the new arrangement

$$\begin{array}{l} \hline \tfrac{1}{3}(\quad\ \ z+\ \ 8)=x \\ \tfrac{1}{3}(-5z+11)=y \quad \text{.................................(7)} \\ \qquad\ \ z \qquad\ \ =z \end{array}$$

* The arrangement (though not the explanation) is in effect that of A. C. Aitken, "Expansion of a certain triple product matrix", *Proc. Roy. Soc. Ed.*, lvii, (1937), p. 172.

All the given equations have now been used, and so we now have the general solution of the equations. z may have any value, and x and y will then have the values given by (7).

4. *General linear equations in integers.*

Now let us take any system of linear equations with integer unknowns and see how the general solution may be got by a like process. For example, let us take the same system (6) as before. As a start we again put a line under the two given equations, and under this line the 3 unknowns whose values we have to get.

As before, we may do the multiplication of any of the rows over the line by any number: and we may do the addition of any multiple of any of the rows over the line to any row other than itself.

But there is one other operation which is a help in the solution. In place of one of the unknowns, say x, we may put a new unknown $u = x - ny$, where n is an integer, and y another unknown. We see that x and y are integers if and only if u and y are integers, and so the new equation we shall get will be equivalent to the old.

But because $ax + by = au + (b + na)y$, the effect of this change on any linear function of $x, y, \ldots$ will be to make the addition of n times the coefficient of x to the coefficient of y: the new coefficient of u is the same as the old coefficient of x.

Now the arrangement (6) may be made in such a way that all the terms in any one unknown are in one column, and, in the same way, all the constant terms in one column. We then get the rule: we may make the addition of any integer multiple of any column of coefficients, other than the constant column, (let us say the column of coefficients of x), to any other column of coefficients, or to the column of constant terms. The effect will be to put a new integer unknown in place of x.

For example, taking the arrangement (6) we may make the addition of the coefficients of -2(x-column) to the y-column, and -3(x-column) to z-column, and $+10$(x-column) to constant column. In place of x we have a new variable u, and we now get the arrangement

$$\begin{array}{r}
u \qquad\qquad\quad = 0 \\
2u - 3y - 5z + 11 = 0 \\
\hline
u - 2y - 3z + 10 = x \\
y \qquad\quad\; = y \\
z \qquad = z
\end{array}$$

By the first equation u may now be dropped everywhere from the equations. Again, by the addition of -2(y-column) to z-column, and 4(y-column) to the constant column, we get

$$\begin{array}{r}
-3v + z - 1 = 0 \\
\hline
-2v + z + 2 = x \\
v - 2z + 4 = y \\
z \quad = z
\end{array}$$

And by the addition of 3(z-column) to the v-column, and z-column to the constant column, we get

$$\begin{array}{r}
w \quad = 0 \\
\hline
v + w + 3 = x \\
-5v - 2w + 2 = y \\
3v + w + 1 = z
\end{array}$$

that is, dropping w by the first equation

$$\left.\begin{aligned} x &= v+3 \\ y &= -5v+2 \\ z &= 3v+1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

and so this is the general solution of the equations (6) in integers.

In the same way, if we have any system of equations, we may, by a number of additions of integer multiples of one column (other than the constant column) to another, get the first equation into the form

$$h_1u = H_1.$$

If then H_1 is not an integer multiple of h_1, clearly the equations have no solution. On the other hand, if H_1 is a multiple of h_1, we have $u = H_1/h_1$, and we may put this into the other equations. Some of these other equations may then take the form $c=0$; if c is not 0, this is impossible, and again the system of equations has no solution; but if this is not so, and $c=0$, then this equation is simply dropped. We may now go on to get the second equation into the form $h_2v = H_2$, and so on, till we come to an impossible equation, or till all the equations have been used up, and we have the general solution.

The number of operations needed to get the general solution in this way is not great; and with care, and by the use of tricks such as the addition of a multiple of one equation to another, it may frequently be made quite small. It may be noted that this way of getting the general solution may be used for the equation in two unknowns, $ax+by+c=0$. It is enough to get the value of x, as that of y is then got from the given equation. When the coefficients are not small, this may be quicker than the process given in part 2, for in that process (but not this one) there are some divisions to be done.

5. *Arithmetical progressions.*

Another way of getting the solution of $ax+by+c=0$ which has its points of interest, and which in the opinion of many persons is simpler than the other ways, is by the use of arithmetical progressions. For a solution of $ax+by+c=0$ is equivalent to getting a term of the A.P., c, $c+b$, ..., $c+by$, ... which is a multiple of a. $(y+1)$ is then the number of the term in the A.P., and $x = -a^{-1}$ (the term in question).

Now the question which we will take for discussion is this: *What is the first term of the* A.P., c, $c+b$, ..., $c+mb$, ... *which is a multiple of a?* In addition we will take a and $b>0$; for any other example (but the unimportant ones, a or $b=0$) may be put in this form by changing if necessary the sign of a, or of b and c.

The solution may be got by using two simple operations. Let $c+bm$ be the desired first term. Then

(i) If the differences between $B>0$ and b, and between C and c, are multiples of a, then clearly

$$C+mB = na \text{ (say) } (m, n \geqq 0) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

is the first term of the A.P., C, $C+B$, ... which is a multiple of a. For example, B, $C+a$ may be the remainders of b, c after division by a.

(ii) But from (9), because B and $a>0$, we see that $-C+na=mB$ is the first term of the A.P., $-C$, $-C+a$, ... which is a multiple of B. This gives a simpler relation for m.

These two operations may be done without difficulty in one step: and by enough steps of that sort we may make the A.P. very simple.

For example, for the solution of $7l-18m=19$ we get the first term $(=7l)$ of the A.P., 19, $19+18, \ldots, 19+18m, \ldots$, which is a multiple of 7. But $19-3.7=-2$, $18-2.7=4$, and so $4m=$ first term $2+7n$ which is a multiple of 4: or because $2-4=-2$, $7-4=3$,

$$3n = \text{first term } 2+4p \text{ which is a multiple of } 3.$$

Clearly $n=2$, $m=4$, $l=13$: and so we have the solution of the given equation.

One last note. If the process is to be frequently used, certain contractions will be made, such as that of writing down only the coefficients in the equations. But these will come so naturally when needed that it seems unnecessary to give an account of them here.

V. E. G.

C. A. B. S.

LEEDS BRANCH.

THREE meetings were held during 1943. At the spring meeting Professor E. C. Stoner of Leeds University gave a paper entitled "The Role of Mathematics in Physics as seen by a Physicist". The questions and discussion afterwards were very interesting.

The summer meeting took the form of a Mathematical Brains Trust, the members of which were Professor Milne, Professor Brodetsky, Miss Pickup, Mr. Montagnon and Mr. Sutton, with Mr. Watts as Question-master. Many of the questions submitted caused lively discussions among the members of the Trust, with a few added comments from the audience.

At the autumn meeting, Mr. Watts, Vice-President, gave a paper on "The Mathematical Curriculum in the Main School", based on the Norwood Report. He also outlined the proposed new syllabus of the Cambridge Local Examinations Syndicate. Keen discussion followed, and many members expressed their views on syllabuses. Mr. Watts proposed that a sub-committee be appointed to discuss fully the geometry syllabus and the whole mathematics syllabus, and seven members were duly elected. This committee's recommendations are to be sent to members before the next meeting in March, when further discussions are to be held and a final report sent to the Mathematical Association.

All the officers, under the Presidency of Miss F. M. Pickup, continue in office for another year. E. HUDSON, *Acting Hon. Secretary.*

1443. With a new English alphabet . . . I should be able to spell t-h-o-u-g-h with two letters, s-h-o-u-l-d with three, and e-n-o-u-g-h with four: nine letters instead of eighteen: a saving of a hundred per cent of my time and my typist's time and the printer's time. . . .—G. B. Shaw, preface to Guild Book edition of *The Miraculous Birth of Language* by R. A. Wilson (1941), p. 21. [Per Prof. E. H. Neville.]

1444. There is a tradition that fiction characters have to be called *something*. Of course writers might call them X 76-4 or Pi R Square. But if we did, all the persons with automobile licenses numbered X 76-4 and all the coolies named Pi Lung Squong would write to us, which heaven forbid.—Sinclair Lewis, "Notice" in *Bethel Merriday* (1940). [Per Prof. E. H. Neville.]

MATHEMATICAL NOTES.

1701. *The limit of* $\Delta^n f(x)/\Delta x^n$.

This note was suggested by Note 1635 in the *Gazette*, XXVI (December, 1942), the present writer having detected similar deficiencies in the usual treatments of the subject. The proof below is essentially different from that of Note 1635 and is of a more general nature.

For proofs of Lemmas 1 and 2 the reader is referred to W. H. Young, *The fundamental theorems of the differential calculus* (Cambridge Tracts, 11). Brief indications of the proofs are given below. Lemma 3 is an identity found necessary to complete the proof and proved by the writer.

In the matter of notation, the fundamental difference formula (3) of Note 1635 is assumed, and the symbol $\exists$ is used throughout to mean " there exists ". In Lemma 2, it will be observed that as the limit point $y=a$ is approached only from the right, as indicated by the notation $y \to a+0$, we need only right-hand derivatives and right-hand continuity at the point $y=a$; but to avoid obscurity of statement, this has not been stated in the lemma. The same remark is obviously appropriate to the main theorem where the lemma is applied. With the hypotheses of the theorem as actually stated, a similar proof establishes the result for the left-hand limit.

Theorem. If $f(x)$ and all derivatives up to order n exist at a point, and if $f(x)$ and the first $(n-1)$ derivatives are continuous at the point, then

$$\exists \lim_{\Delta x \to +0} \frac{\Delta^n f(x)}{\Delta x^n} = f^{(n)}(x).$$

Lemma 1. If (i) $F(y), G(y) \to 0$ as $y \to a+0$,

(ii) $\exists \lim_{y \to a+0} \dfrac{F'(y)}{G'(y)} = l$,

(iii) $G'(y)$ maintains a constant sign in the interval $a<y<a+\delta, \quad \delta>0$,

then $$\exists \lim_{y \to a+0} \frac{F(y)}{G(y)} = l.$$

For any $m>l$, we can find an interval $a<y<a+\delta, \quad \delta>0$,

in which $$F'(y)/G'(y)<m.$$

By considering $$\phi(y) = F(y) - m \,.\, G(y),$$

we prove that $F(y)/G(y)<m$ in the same interval.

A corresponding result is obtained for any $n<l$.

Lemma 2. If (i) $F(y)$, $G(y)$ and all derivatives up to order $(n-1)$ are continuous at $y=a$ and vanish at $y=a$,

(ii) $\exists F^{(n)}(a), \quad G^{(n)}(a), \quad$ and $\quad G^{(n)}(a) \neq 0$,

then $$\exists \lim_{y \to a+0} \frac{F(y)}{G(y)} = \frac{F^{(n)}(a)}{G^{(n)}(a)}.$$

The proof consists of two parts. From conditions (i) and (ii) and the definition of a derivative, we obtain

$$\exists \lim_{y \to a+0} \frac{F^{(n-1)}(y)}{G^{(n-1)}(y)} = \frac{F^{(n)}(a)}{G^{(n)}(a)}.$$

We then show that Lemma 1 may be applied successively to the pairs

$$(F^{(n-1)}, \quad G^{(n-1)}), \quad (F^{(n-2)}, \quad G^{(n-2)}), \dots (F', G'),$$

from which the result follows.

Lemma 3. If $E(k)$ is defined for any positive integer n by

$$E(k) \equiv \sum_{r=0}^{n} (-)^r \binom{n}{r} (n-r)^k$$
$$\equiv n^k - \binom{n}{1}(n-1)^k + \ldots + (-)^r \binom{n}{r}(n-r)^k + \ldots$$

then $$E(k)=0 \text{ for } 0 \leqslant k < n$$
and $$E(n)=n!$$

Proof. The case $k=0$ is trivial since it gives

$$E(0)=(1-1)^n=0.$$

Consider the identity

$$y=(x-1)^n \equiv x^n - \binom{n}{1}x^{n-1} + \ldots + (-)^r \binom{n}{r} x^{n-r} + \ldots + (-1)^n.$$

We apply the operation $x\dfrac{d}{dx}$ to y, k times in succession, the result being denoted by $\left(x\dfrac{d}{dx}\right)^k y$. Finally we put $x=1$.

Using the second expression for y, we obtain

$$\left[\left(x\frac{d}{dx}\right)^k y\right]_{x=1} = \left[\sum_{r=0}^{n} (-)^r \binom{n}{r} (n-r)^k x^{n-r}\right]_{x=1}$$
$$=E(k).$$

But $$x\frac{dy}{dx}=nx(x-1)^{n-1},$$

$$\left(x\frac{d}{dx}\right)^2 y = nx(x-1)^{n-1} + n(n-1)x^2(x-1)^{n-2},$$

and so on.

Evidently for $0<k<n$, a factor $(x-1)$ remains and so the value of the expression is 0 when $x=1$.

If $k=n$, there is one term which does not contain $(x-1)$, namely $n!\,x^n$, and this has the value $n!$ when $x=1$.

Hence the result $E(k)=0$, $0 \leqslant k<n$; $E(n)=n!$.

Proof of the theorem.

Consider $$F(y) \equiv f(x+ny) - \binom{n}{1} f\{x+(n-1)y\} + \ldots$$
$$+(-)^r \binom{n}{r} f\{x+(n-r)y\} + \ldots + (-)^n f(x),$$
$$G(y) \equiv y^n.$$

Then $G(y)$ satisfies the conditions (i) and (ii) of lemma 2 at $y=0$.

If $0 \leqslant k < n$,

$$F^{(k)}(y) = n^k . f^{(k)}(x+ny) + \ldots + (-)^r \binom{n}{r} (n-r)^k f^{(k)}\{x+(n-r)y\} + \ldots,$$

and hence $F, F', \ldots F^{(n-1)}$ are continuous at $y=0$.

$$F^{(k)}(0) = n^k f^{(k)}(x) + \ldots + (-)^r \binom{n}{r} (n-r)^k f^{(k)}(x) + \ldots$$
$$=E(k) . f^{(k)}(x)$$
$$=0, \text{ by lemma 3.}$$

Also, $\exists\, F^{(n)}(0)=E(n)\,.\,f^{(n)}(x)=n!\,f^{(n)}(x)$, by lemma 3.
Thus $F(y)$ satisfies the conditions (i) and (ii) of lemma 2.
Applying lemma 2, we obtain

$$\exists \lim_{y\to+0}\frac{F(y)}{G(y)}=\frac{n!\,f^{(n)}(x)}{n!}=f^{(n)}(x).$$

Writing now $y=\Delta x$, and observing that $F(\Delta x)\equiv\Delta^n f(x)$, we have

$$\exists \lim_{\Delta x\to+0}\frac{\Delta^n f(x)}{\Delta x^n}=f^{(n)}(x).$$

P. GANT.

1702. *A note on quadrilaterals.*

I. *Cyclic quadrilateral.*

Let $ABCD$ be a cyclic quadrilateral whose diagonals intersect at O, and let $AB=u$, $BC=v$, $CD=x$, $DA=y$, $DB=d_1$, $AC=d_2$. Then

(i) if we interchange u and v (or x and y) and keep $\angle ABC$ and $\angle ADC$ fixed, or
(ii) if we interchange v and x (or u and y) and keep $\angle DAB$ and $\angle BCD$ fixed,

we get two more cyclic quadrilaterals of the same area, Q, and with the same circumcircle ; let R be its radius.

Also, in each of the two new quadrilaterals the angle between the sides u and x will be equal to $\angle AOB$, and *one* of the diagonals will be d_1 or d_2, while the other will be d_3 given by

$$\begin{aligned} d_1(vx+uy)&=d_2(uv+xy)=d_3(ux+vy)\\ &=d_1d_2d_3=\sqrt{\{(vx+uy)(uv+xy)(ux+vy)\}}\\ &=4RQ\\ &=8R^3\sin DAB\sin ABC\sin AOB.\end{aligned}$$

It is not without interest to notice the limiting case, obtained by putting, say, $y=0$, when $d_1=u$, $d_2=x$, $d_3=v$, and we get the familiar result

$$abc=4R\Delta=8R^3\sin A\sin B\sin C.$$

II. *Cyclic and circumscribable quadrilateral.*

Let $ABCD$ be such a quadrilateral, and let $AB=u$, etc., as above. Also let r be the radius of the inscribed circle, and let

$$t_1=\tan\tfrac12 DAB=\cot\tfrac12 BCD,\quad t_2=\tan\tfrac12 ABC=\cot\tfrac12 ADC.$$

(For convenience we will assume that $\angle DAB$ and $\angle ABC$ are the two acute angles.) Then, since

$$u=r(\cot\tfrac12 DAB+\cot\tfrac12 ABC),\text{ etc.,}$$

we get $\quad u/(t_1+t_2)=v/t_1(1+t_1t_2)=x/t_1t_2(t_1+t_2)=y/t_2(1+t_1t_2)=r/t_1t_2,$

and the area, Q, of the quadrilateral is given by

$$r(u+x)=r(v+y)=r^2(t_1+t_2)(1+t_1t_2)/t_1t_2.$$

If we now choose l, m n so that

$$l:m:n=(1+t_1)(1+t_2):(1-t_1)(1+t_2):(1-t_1)(1-t_2),$$

that is, replace t_1 by $(l-m)/(l+m)$ and t_2 by $(m-n)/(m+n)$, we get

$$\begin{aligned} u/(l+m)(m+n)(l-n)&=v/(l-m)(m+n)(l+n)\\ &=x/(l-m)(m-n)(l-n)\\ &=y/(l+m)(m-n)(l+n)\\ &=2mr/(l^2-m^2)(m^2-n^2)=K,\text{ say.}\end{aligned}$$

Further, using the results given in § 1, we get

$$K=\frac{d_1(l^2+m^2)}{2\rho(l^2-m^2)}=\frac{d_2(m^2+n^2)}{2\rho(m^2-n^2)}=\frac{d_3(l^2+n^2)}{2\rho(l^2-n^2)}=\frac{R}{\rho},$$

where $$2\rho^2=(l^2+m^2)(m^2+n^2)(n^2+l^2),$$

and the condition for rational (or integral) sides, diagonals and area is given by the above formula provided l, m, n and ρ are all rational (or integral). The condition that ρ shall be rational is, of course, the only difficult one to satisfy.

The general formula given by Mr. Peacock in his interesting Note 1634 (*Gazette*, XXVI, December 1942) corresponds to the equation

$$l/n=(7m^4+10m^2n^2-n^4)/(7n^4+10m^2n^2-m^4)$$

in which case

$$\rho=\frac{n(m^2+n^2)(5m^4+6m^2n^2+5n^4)(m^4+14m^2n^2+n^4)}{(7n^4+10m^2n^2-m^4)^2},$$

and the numerical example he gives corresponds to the values $l=11$, $m=3$, $n=1$, but I have obtained several other examples, not included in his formula, of which the simplest are

(i) $l=11$, $m=3$, $n=2$, giving $AB=315$, $BC=260$, $CD=36$, $DA=91$, $AC=125$, $BD=280$;

(ii) $l=12$, $m=5$, $n=3$;

(iii) $l=38$, $m=6$, $n=1$;

(iv) $l=46$, $m=17$, $n=9$;

(v) $l=82$, $m=11$, $n=2$;

(vi) $l=189$, $m=43$, $n=7$.

Notes. 1. If, as in (vi) above, two of the letters l, m, n have a common factor, this can be allowed for when using the general formula.

2. If u and v are interchanged, or if v and x are interchanged, as in I., it can be seen that a circle can now be drawn to touch the four sides of the quadrilateral externally.

III. *A special quadrilateral.*

Every " kite " quadrilateral formed by two congruent right-angled triangles with a common hypotenuse, and with rational sides, is a cyclic and circumscribable quadrilateral with rational sides, diagonals and area.

All such quadrilaterals with *integral* sides, etc., are given by the general formulae

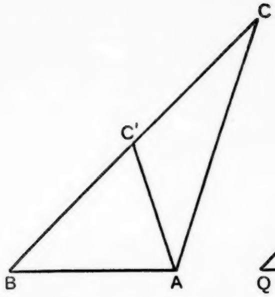
$$AB=AD=k(l^2+m^2)(l^2-m^2),$$
$$BC=CD=2klm(l^2+m^2),$$
$$AC=k(l^2+m^2)^2,\quad BD=4klm(l^2-m^2).$$

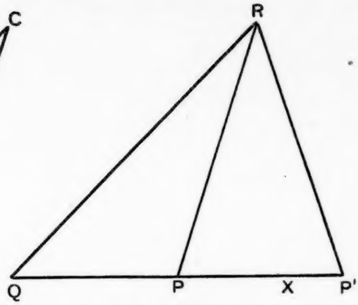
D. F. FERGUSON.

1703. *Solution of triangles: a little-noted ambiguous case.*

If two suitable data concerning a triangle ABC are given it is in general possible to determine the shape of the triangle. The problem is determinate if (i) A, B, (ii) $a:b:c$, (iii) $a:b$, C, or (iv) $a:b$, B ($b/a>1$) are given. In these cases the measure of one side is sufficient to determine the scale of the triangle. The data $a:b$, B lead to ambiguity of a double kind if $b/a<1$ (b necessarily $\geqslant a\sin B$).

From the relation $\sin A=a\sin B/b$ two values of the angle A are obtained both of which are admissable if $b/a<1$, giving two values of the ratio $a:b:c$.





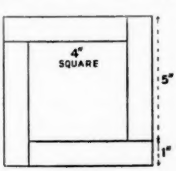
If the measure of the side a or of the side b is now given there are two values of the side c and a and b are the same for both alternatives. This is the case usually discussed in textbooks on elementary trigonometry. It will be referred to as Case I. If, on the other hand, the measure of the side c is given there are two values of a and b which are different for the two alternatives. This will be referred to as Case II.

The duality of the two cases is exemplified in one method of solving the following problem: "A particle is projected from a point B with speed V along a line BC and at the same time a particle is projected from a point A with speed v ($V > v > V \sin ABC$). Find the directions in which the second particle may be projected to strike the first."

The velocity diagram is completed as follows. Let QR represent V and draw QX parallel to BA, the direction of the vector $V - v$. With centre R and radius v draw an arc to cut QX in P and P'. PR, $P'R$ give the two possible directions of projection. This is ambiguous Case I. The space diagram can now be completed by drawing through A parallels to PR, $P'R$ meeting AC in C, C'. This is ambiguous Case II. RAYMOND SMART.

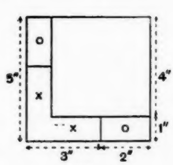
1704. *Solution of a geometrical problem.* (*Note* 1672).

The blocks being placed on a table, the 3″ and 5″ cubes are divided horizontally into (three and five) square boards 1″ thick. Surround the 4″ cube by four of the 5″ square boards covering its vertical faces and forming a 6″ square on the table (Fig. A).



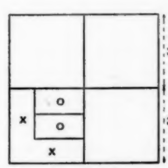
A

Base of the 6″ cube.



B

Dissection of the 5″ square.



C

Top of the 6″ cube.

We have as it were a tower whose base is a 6″ square and whose outer walls are 5″ high; at the top of the tower is a depression 1″ deep (since the central block is only 4″ high) which we fill by a 4″ square cut out of the last 5″ square board (Fig. B). We now have a solid rectangular block 6″ × 6″ × 5″, and we

need only a 6″ square of thickness 1″ to complete the 6″ cube. The three square boards derived from the 3″ cube form three quarters of such a square. If the L-shaped border of Fig. B is divided into three parts by cuts 2″ from the ends of the legs of the L, the parts can be arranged in a square (Fig. C) so as to form the last quarter. H. W. R.

1705. *The framework in n-dimensions.*

§ 1. The external actions on a rigid body in equilibrium in n-dimensional space must satisfy

$$\frac{n!}{2!(n-2)!}+n=\binom{n}{2}+n \quad \text{.............................(1)}$$

conditions. For, if we imagine the body in relation to n co-ordinate axes, the external actions must have no resultant parallel to each axis and they must provide no couple parallel to any of the $\binom{n}{2}$ planes corresponding to the possible pairs of axes.

If the rigid body is composed of a number of joints linked together, if, that is, it is a *framework*, then at each joint there will be n equilibrium conditions to be fulfilled. If there are j joints in all, the number of equations to be satisfied would appear to be nj. As the actions on the body as a whole have to fulfil $\binom{n}{2}+n$ equations, however, the number of independent equations to be satisfied by the external actions presumed applied at the joints and the actions induced in the links must be

$$nj-\binom{n}{2}-n. \quad \text{.....................................(2)}$$

This must therefore be the smallest number of links between the joints. Fewer would mean that all the joints were not definitely related in space to the others: more would mean unnecessary or redundant relationships.

(2) can better be written

$$nj-\tfrac{1}{2}n(n+1), \quad \text{.....................................(3)}$$

or

$$(j-1)+(j-2)+(j-3)+\ldots+(j-n). \quad \text{.......................(4)}$$

The structural engineer will recognise the formulae obtained by putting $n=2$ and $n=3$ in (3). He may not have noticed the alternative expressions given by (4), however. These special cases are noted in the table herewith.

§ 2. In n-dimensional space the stable n-dimensional framework with the smallest number of joints must be that in which

2.1. at each joint there meets the minimum number of links for equilibrium, which is n.

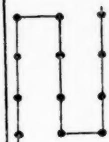
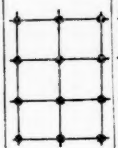
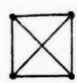
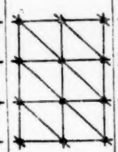
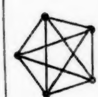
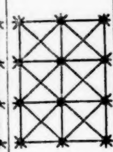
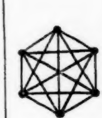
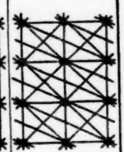
2.2. every joint is linked directly to every other joint.

It follows that the smallest n-dimensional framework is a symmetrical one with $(n+1)$ joints. Simple examples are given in the table.

§ 3. The average number of links meeting at a joint of an n-dimensional framework is, from (3),

$$2n-\frac{n\,.\,(n+1)}{j}. \quad \text{.....................................(5)}$$

Presuming the infinite just stable framework of n-dimensional space to be symmetrical, it must, putting $j=\infty$ in (5), have $2n$ links meeting at each joint. Two-dimensional representations of typical portions of the simplest of these frameworks are given in the table.

Number of Dimensions. (n)	Number of links in just-stable framework with j joints. from formula (3)	from formula (4)	Two-dimensional representation of n-dimensional framework with minimum number of joints.	Two-dimensional representation of part of infinite n-dimensional framework.
1	$j-1$	$(j-1)$		
2	$2j-3$	$(j-1)+(j-2)$		
3	$3j-6$	$(j-1)+(j-2)+(j-3)$		
4	$4j-10$	$(j-1)+(j-2)+(j-3)+(j-4)$		
5	$5j-15$	$(j-1)+(j-2)+(j-3)+(j-4)+(j-5)$		

H. ROXBEE COX.

1706. *Motion when mass is changing.*

On reading Mr. Ramsey's article in the *Gazette*, October 1942, I seemed to detect something strange, and then looking at his article in the *Gazette* for July 1941 the results given were not quite what might be expected.

Consider his example (i), *Gazette*, XXV, p. 141 (July 1941).

It is fairly obvious from the rough diagram (Fig. 1) that the drop of water of

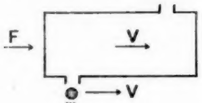


FIG. 1.

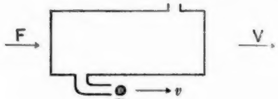


FIG. 2.

mass m, which leaks from the engine, exerts no force on the engine horizontally. The changing mass, however, will alter the motion produced by F. Applying Mr. Ramsey's method,

$$\text{at time } t, \text{ the momentum} = MV;$$
$$\text{at time } t+\delta t, \text{ the momentum} = (M - m\delta t)(V + \delta V).$$

I do not think, however, that we can add the last term of the equation given on p. 141, for indeed the water has separated from the body under consideration. We have, therefore,

$$\begin{aligned}\text{change in momentum} &= (M - m\delta t)(V + \delta V) - MV\\ &= M\delta V - mV\delta t, \text{ to the first order.}\end{aligned}$$

Equating the momentum change to the impulse of the force we have

$$M(dV/dt) - mV = F,$$

a result which is not independent of m. Now $m = -dM/dt$, and so

$$M(dV/dt) + (dM/dt)V = F$$

or

$$d(MV)/dt = F.$$

We consider now example (ii). In order to give physical reality to the problem, we shall have to suppose that a curved tube projects into the water (Fig. 2). Such a tube without drawing up water will affect the motion, but we will suppose that its retarding action has been included in the frictional forces. The change in momentum of the collected water moving initially with velocity v can only be brought about by a force F' whose reaction opposes the motion of the engine, and

$$F' = m(V - v).$$

The mass of the water in the boiler will alter, and, consequently,

$$\text{change of momentum in time } \delta t = (M + m\delta t)(V + \delta V) - MV$$

and as before

$$M\delta V + mV\delta t = F\,.\,\delta t - F'\,.\,\delta t,$$

and so, since $m = dM/dt$, we have

$$d(MV)/dt = F - F'.$$

It is difficult to agree with the solution to example (iii) or to the facts of the problem (Fig. 3).

We should note at once, following the law of conservation of mass, that

if mass m of fuel is " consumed ", m will also be the mass of the products. The change in mass can only be due to the ejected materials. (We neglect any change in mass due to combination with oxygen.) This example then reduces to example (i) and the solution is the same.

I do not think there is anything really new in the example on p. 116 of the *Gazette* for October 1942. It seems possible to treat the problem in a similar way and to suppose that a curved tube (Fig. 4) is used for the collection of the oxygen. Here $V>v$ and the rate of injection is $m'-m$ units of mass per second. This injection of oxygen will cause a backward force on the engine given by

$$F_1=(m'-m)(V-v).$$

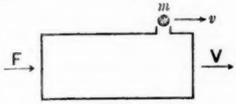


FIG. 3.

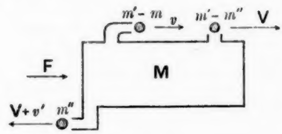


FIG. 4.

The ejection of ashes causes a force F_2 in the direction of motion and $F_2=m''v'$, since m'' units of mass are ejected per second. The change in momentum in time δt is

$$[M+(m'-m)\delta t-m''\delta t-(m'-m'')\delta t](V+\delta V)-MV,$$

or to the first order

$$(M-m)(V+\delta V)-MV,$$

and as $dM/dt=-m$, we have

$$d(MV)/dt=F-F_1+F_2.$$

In these equations the motion of one body only has been considered. To extend the momentum equation to a system of bodies, we have to note that, when reactions are set up in a system by the transfer of mass from one body to another, it is, as a consequence of the law of conservation of linear momentum, possible to write

$$\Sigma d(MV)/dt=0,$$

where M refers to the mass of each separate body and V to its velocity in the direction under consideration. If, in addition, $F_1, F_2, F_3, \ldots$ are the components of external forces or constraints in the same direction,

$$\Sigma d(MV)/dt=\Sigma F.$$

This equation can be verified by considering the motion of each body separately.

In the example dealing with the injection of water into the engine,

$$d(MV)/dt=F-F',$$

but, if we suppose the curved tube to project into a freely moving mass of water m', then

$$F'=d(m'v)/dt.$$

In this case

$$d(MV)/dt=F-d(m'v)/dt,$$

or

$$\Sigma d(MV)/dt=F.$$

Such an equation appears to give correct solutions for all the usual examples on motion with changing mass.

H. LANGDON-DAVIES.

1707. ***A proof by induction of a theorem on plane curves.***

Mathematical induction is a method I have seldom noticed in works on plane geometry. Having recently used this method, I would be interested to hear of further applications. I send the following note, therefore, both as one example and also, chiefly, as a quest for information from readers of the *Gazette* concerning geometrical problems which can be solved in this way.

If $A_{ij}(i, j=1, 2, \dots, n+1, n+2, i\neq j, A_{ij}\equiv A_{ji})$ *are the* $\frac{1}{2}(n+2)(n+1)$ *points of intersection of* $n+2$ *tangents* a_i $(i=1, 2, \dots, n+2)$ *to a conic* Σ *and if* PP_1P_2 *is a triangle circumscribing this conic, then the curves* C^{n+1}, *of order* $n+1$, *which pass through the points* A_{ij} *and through* P *and* P_1 *necessarily pass through* P_2.

When I first required a proof of this result Dr. J. Bronowski indicated to me the following method of induction and the proof now given is the direct outcome of that suggestion. I have since found the theorem elegantly proved algebraically by W. L. Edge in " Some Special Nets of Quadrics in Four-Dimensional Space ", *Acta Mathematica*, 66 (1936), 276.

For lack of a direct geometrical method, we prove it now by induction.

Case $n=1$. The result (if two triangles circumscribe a conic then their vertices lie on another conic) is classical.

The results for $n=2$ and $n=3$ are not essential in a proof by induction but here they are not without interest and help to picture the proof in the general case. When $n=2$ we have to show that the 9 points (all A_{ij} and P, P_1, P_2) are " associated points " on a pencil of cubic curves, *i.e.* that there are two independent cubics $C_1{}^3=0$ and $C_2{}^3=0$ through the 9 points, so that any cubic with the equation $C_1{}^3+\lambda C_2{}^3=0$, λ being any constant, passes through the same 9 points. One such cubic consists of the tangent a_4 plus the conic (see case $n=1$) through the remaining 6 points. Another such cubic is formed by a_1 plus the conic through the remaining 6 points. These are two independent cubics $C_1{}^3=0$ and $C_2{}^3=0$ through the 9 points, which are therefore associated. This case and, in effect, the foregoing proof are given in Baker, *Principles of Geometry*, Cambridge, Vol. III (1934), 217.

For $n=3$, A_{ij}, P and P_1 are in all 12 points. Since the general equation of the fourth degree in x and y contains 15 terms, a quartic curve is uniquely determined by 14 independent points. So our system of quartics has two degrees of freedom, and we have, therefore, to prove that the ∞^2 system (or " net ") of quartic curves C^4 through all A_{ij}, P and P_1 must pass through P_2. One ∞^1 of such quartics consists of a_5 plus the pencil of cubics (see case $n=2$) through the remaining 9 points. Another ∞^1 system consists of a_1 plus the pencil of cubics through the remaining 9 points. These two ∞^1 systems have in common only one quartic, namely a_1, a_5 and the residual conic (see case $n=1$) through the remaining 6 points. Hence the two systems determine an ∞^2 of quartic curves with the required property.

General Case. We assume that we have proved the result up to and including the case $n=m-1$. It is easily proved that the points A_{ij} $(i, j=1, 2, \dots, m+2)$, P and P_1 impose independent conditions on the curves of order $m+1$ to contain them. For, one such curve consists of all the lines a_i except a_1, a_2 and a_3, together with a residual conic which passes through A_{23}, A_{31}, A_{12}, P and P_1. It is clear that a conic through 4 of these points does not necessarily pass through the fifth. Thus, not all curves of order $m+1$ through A_{ij} and P pass through P_1. Similarly not all such curves through P, P_1 and all A_{ij} except one point (say A_{12}) pass through that remaining point. Hence, the points A_{ij}, P and P_1 are independent in regard to the curves of order $m+1$ through them. There are then $\frac{1}{2}(m+2)(m+1)+2$

independent points through which the curves C^{m+1} must pass. Since the general equation of the $(m+1)$th degree in x and y contains $\frac{1}{2}(m+3)(m+2)$ terms, a curve of order $m+1$ is uniquely determined by $\frac{1}{2}(m+3)(m+2)-1$, *i.e.* $\frac{1}{2}(m+1)(m+4)$ independent conditions. The present system, therefore, has a freedom of $\frac{1}{2}(m+1)(m+4)-\frac{1}{2}(m+2)(m+1)-2$, *i.e.* $m-1$. We have now to prove that the ∞^{m-1} of curves C^{m+1} of order $m+1$ through all A_{ij}, P and P_1 pass through P_2. One ∞^{m-2} system of such C^{m+1} consists of a_{m+2} plus the ∞^{m-2} system (see case $n=m-1$) of C^m. Another ∞^{m-2} system consists of a_1 plus similarly an ∞^{m-2} system of C^m. These two ∞^{m-2} systems have in common an ∞^{m-3} of C^{m+1} consisting of a_1, a_{m+2} and an ∞^{m-3} system of C^{m-1} (see case $n=m-2$). Hence, since $2(m-2)-(m-3)=m-1$, the two systems determine an ∞^{m-1} of curves C^{m+1} which pass through all A_{ij}, through P and P_1 and through P_2. If, therefore, the result is true up to the case $n=m-1$ it is also true up to the case $n=m$. But it is true for $n=1$, therefore for $n=2$, etc., and so true for general n.

I hope to publish elsewhere the applications of this theorem which first necessitated my proving it. There may be, however, some interest in showing here that the result leads readily to that generalisation of Poncelet's porism for which it was required and given (*loc. cit.*) by Dr. Edge. Poncelet's porism states that if two conics Σ and C^2 are such that there exists one triangle which is circumscribed to Σ and inscribed to C^2 then an infinity of such triangles exists. This result is obvious from case $n=1$ above. For, let $A_{23}A_{31}A_{12}$ be any triangle circumscribed to Σ and inscribed to C^2. Take any point P on C^2 and from P draw the tangents t_1 and t_2 to Σ. Let t_1 meet C^2 again in P_1; from P_1 draw the other tangent to Σ to meet t_2 in P^2. As above, P_2 lies on C^2. Hence, for every point P on C^2 there is a triangle PP_1P_2 which is circumscribed to Σ and inscribed to C^2.

The case $n=2$ yields the porism: if a conic Σ and a cubic C^3 are such that there exists one complete quadrilateral which is circumscribed to Σ and inscribed to C^3 then an infinity of such quadrilaterals exists. For, let $A_{ij}(i, j=1, 2, 3, 4)$ be the 6 vertices of any quadrilateral circumscribed to Σ and inscribed to C^3. Take any point P on C^3 and draw the tangents t_1 and t_2 from P to Σ. Let t_1 meet C^3 again in P_1 and P_1'; from P_1 draw the other tangent to Σ to meet t_2 in P_2. As above, P_2 lies on C^3. From P_1', also, draw the other tangent to Σ to meet t_2 in P_2'. Similarly, P_2' lies on C^3. If, further, the other tangents from P_1 and P_1' meet in P_{12}, then as above $P_1P_1'P_{12}$ is a triangle circumscribing Σ. C^3 passes through P_1 and P_1' and so through P_{12}. Hence, for every P on C^3 there is a complete quadrilateral (namely $PP_1P_2P_1'P_2'P_{12}$) which is circumscribed to Σ and inscribed to C^3.

We can extend this argument immediately to the general case. Suppose that a conic Σ and a curve C^{n+1} of order $n+1$ are such that there exists one $(n+2)$-gram which is circumscribed to Σ and inscribed to C^{n+1}. From any point P on C^{n+1} draw a tangent to Σ. Let P_1P_1' be any two intersections of this tangent with C^{n+1}. Then we can show, as above, that the other tangents to Σ from P_1 and P_1' meet in a point lying on C^{n+1}. Thus the $n+1$ tangents to Σ from the $n+1$ points in which any tangent to Σ meets C^{n+1} are such that the intersection of any pair of them lies on C^{n+1}. Hence, for every point P on C^{n+1} there is one $(n+2)$-gram which is circumscribed to Σ and inscribed to C^{n+1}. For the general case, therefore, we have the porism: if a conic Σ and a curve C^{n+1} of order $n+1$ are such that there exists one $(n+2)$-gram which is circumscribed to Σ and inscribed to C^{n+1} then an infinity of such $(n+2)$-grams exists.

The porism when $n=3$ would, I believe, have been of special interest to the practitioners of magic in England during Tudor times, for the pentagram

formed part of their equipment and indeed the science of magic was believed to depend on it. Perhaps Dr. Dee (Fellow of Trinity) would have achieved even greater fame had he been aware of Lüroth's porism: if a quartic curve is circumscribed to a pentagram then it is circumscribed to an infinity of pentagrams. No doubt he would have made use of the fact that all such pentagrams circumscribe the same conic. R. JONES.

1708. *Mean values and Simpson's Rule.*

The following ideas are far from new but are, apparently, still unknown to many. To economise space I have abbreviated where possible.

The word "average" is commonly understood to mean *sum/number* (*i.e.* Arithmetic Mean): it is used in this sense in the following.

The word "mean" requires definition but its significance is generally clear in practical cases. For example:

Mean speed × total running time gives distance run,
Mean cross-section × total length gives volume,
Mean force × total distance gives work done,
Mean acceleration × time of observation gives change in velocity,
Mean "Tons per inch immersion" of ship × draught gives displacement and, graphically,
Mean height of a curve × total base length gives area under curve.

This last statement gives the mathematical definition of Mean Ordinate.

How can this mean be found for a variable quantity? It is obviously not the "average" of the first and last, or even of the first, last and middle values—*e.g.* train's mean velocity between two stations. It is an easy step to "the more values you can make use of the better answer you will get".

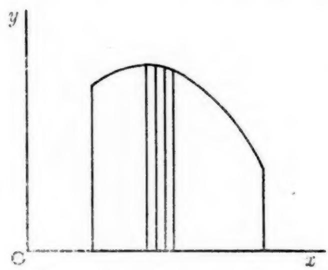
In most cases, however, a remarkably accurate answer can be obtained by a device. Simpson's First Rule ("1, 4, 1") provides one of the simplest and most effective devices. It depends on the fact that the Mean Ordinate of the curve Ax^3+Bx^2+Cx+D (hence of parabola, which is usually the only curve mentioned in this connection) is exactly

$$\frac{y_F+4y_M+y_L}{1+4+1},$$

F, M, L signifying first, middle, last.

Diagrammatically, therefore, the True Mean Height of such a curve is the "average" of the ordinates shown—hence the *only fact to be memorised is* "1, 4, 1".

The extension to "1, 4, 2, ... 2, 4, 1" is obvious.



Simpson's First Rule may, therefore, be stated thus :

If an odd number of measurements of any quantity are taken *at regular intervals* of time, distance or other independent variable (see example below), then the Mean Value of these quantities is the " average " of the first, last, four each of the even numbered ones and two each of the remainder.

If the working is arranged in vertical columns (the most practical way) then the Mean Value is the result of dividing the total of the " products " column by that of the " multipliers " column. If the numbers are cumbersome, much labour will be saved by using Half-Multipliers : the same process then gives the same result—a fact seldom appreciated.

All worked examples should be set in the form : " Find the Mean Value of ... and (if required) *hence* the ... ". The following example illustrates the points mentioned.

Example. Assuming the law $pv=120$, where p is in lb./ft.^{2} and v is in ft.^{3}, find the Mean Pressure between $v=2$ and $v=6$. Hence find the energy expended during the expansion.

v	p	Multiplier	Product
2	60	$\frac{1}{2}$	30
3	40	2	80
4	30	1	30
5	24	2	48
6	20	$\frac{1}{2}$	10
		6)	198
Mean Pressure			33 lb./ft.^{2}

(Integration gives 32·96.)

In this example it is not obvious what "mean" means!—but, if, as indicated in third paragraph of this article, we multiply by the *total* change in volume *not forgetting the units* we get 132 ft.-lb.—*i.e.* energy—which seems to me at any rate no worse than asking the simpler minded to accept the fact than an *area* is energy.

Other " Rules for approximate integration " can be treated in an exactly similar manner—the total memory work being reduced, for example, to :

Simpson's Second (or " three-eighths ") Rule - -	1, 3, 3, 1
Weddle's Rule - - - - - - - -	1, 5, 1, 6, 1, 5, 1
Dufton's Rule - - - - - - - -	0, 1, 0, 0, 1, 0
The " Five, Eight, Minus One " Rule - - -	5, 8, -1

The last mentioned seems little known. It is exact for a parabola and is used thus : To find the Mean Value of *two* quantities when a *third* of the series (equally spaced) is known, use 5, 8 and -1 as multipliers and divide the sum of the products by $5+8-1$, *i.e.* 12.

The five rules mentioned, if applied (with the minimum possible number of ordinates in each case) to find the Mean Value of $\sin x$ between 60° and 90°, give the following results respectively, using four-figure tables : 0·9549, 0·9549, 0·9549, 0·9540 and 0·9523 or 0·9553 depending on whether 30° or 120° is used for the third ordinate. The value by integration is 0·9549. G. A. C.

1445. Newton, Maxwell, Einstein—all these were true adventurers. They wanted to see things that no man had ever seen before. But whereas the new things that Einstein saw are far removed from the world as we know it, the abstruse calculations of Newton and Maxwell proved intensely practical. And alone of all the world's mathematicians Maxwell can claim to have founded a great industry.—A. W. Haslett, *Radio round the World*, p. 5. [Per Mr. F. W. Kellaway.]

REVIEW.

Copernicus. By Sir Harold Spencer Jones. Pp. 32. 1s. 6d. The Selby Lecture, 1943. (University of Wales)

The fifth in the series of lectures founded to commemorate the association of the late Professor A. L. Selby with the University College of South Wales and Monmouthshire was delivered by the Astronomer-Royal at Cardiff in May, 1943. His subject was "Copernicus", and the lecture has been published in the form of an attractive brochure.

The author handles his material with sympathy and critical appreciation throughout, and sketches the background of Aristotelian belief against which the achievement of Copernicus must be viewed.

The statement that Copernicus "was a Pole by birth and a Slav by ancestry" would appear to settle the somewhat vexed question of his nationality.

In 1543, shortly before his death, a printed copy of his celebrated *De Revolutionibus Orbium Coelestium* was placed in his hands, and the publication of this treatise marked one of the most significant changes of outlook which has ever occurred in the history of scientific thought.

It has been remarked by P. M. Dirac that there are two general principles, not perhaps unconnected, to which the physical scientist may appeal when he is in doubt. One is the "Principle of Simplicity", and the other the "Principle of Beauty". In seeking to improve upon the Ptolemaic system of geocentric spheres with its elaborate mechanism of eccentric circles and eipcycles Copernicus appealed to the former principle. Though he was unable to emancipate himself from the belief in the "perfection" of the circular orbit, yet, by placing the Sun at the centre of the Universe he succeeded in reducing the number of such circles from about 80 to 34.

Conscious of the limitations of his system, he nevertheless believed his theory to be, not merely a mathematical device, but physically true, and he made a genuine attempt to meet the physical objections raised against it. He argued in particular "that gravity was not a property peculiar to the Earth, but was a universal force".

From the dimensions of the retrograde arcs of the planets he was able to determine with very fair accuracy the relative distances of the planets from the Sun, and he was the first to relate the phenomenon of precession with a motion of the axis of the Earth. He was even prepared to accept the implication of an infinite universe, though he declined definitely to commit himself.

It was largely through the agency of Thomas Digges (1546–1595), who published an account of the Copernican system in English, that the new cosmology found a readier acceptance in this country than on the Continent. In 1616, when Galileo was indicted by the Catholic Church, the *De Revolutionibus* was placed on the Papal "Index" and declared heretical.

The cause of this dispute now appears remote, for the revolution of the Earth about the Sun and *vice versa* represent but two aspects of one fundamental truth, and it is certain that Copernicus himself realised the essentially relative nature of the celestial motions. "For every apparent change of position is due", he wrote, "either to a motion of the object observed, or to a motion of the observer, or to unequal changes in the position of both."

While some may perhaps feel that the remarkable faculty which Copernicus displayed for anticipating the trends of present-day belief scarcely receives sufficient emphasis in the booklet under review, yet a debt of gratitude is due to the Astronomer-Royal for an informative, concise and felicitous account of the life and work of a most outstanding pioneer in the search for truth.

T. A. B.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS, GLASGOW

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